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Multiple complex and multiple real soliton solutions for the integrable sine-Gordon equation

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ABSTRACT

In this work, we study the integrable sine-Gordon equation which is related to the short pulse equation (SPE). We show that this equation passes the Painlevé test to confirm its integrability. We introduce new complex forms of the simplified Hirota's method to formally derive multiple complex soliton solutions for this equation. Our results show that the complex simplified Hirota's method explicitly constructs new multiple complex soliton solutions for more integrable equations.

1. Introduction

The sine-Gordon equation

$$u_{xt} + \sin u = 0, \tag{1}$$

is a nonlinear hyperbolic partial differential equation (PDE) involving the d'Alembert operator u_{xt} and the sine of the unknown function $u(x, t)$. The sine-Gordon equation appeared first in the study of differential geometry of surfaces [1–16] with Gaussian curvature $K = -1$. It also appears in a number of physical applications, including applications in relativistic field theory, Josephson junctions or mechanical transmission lines [17–37]. It also appears in many scientific fields such as the motion of a rigid pendula attached to a stretched wire [4–8], solid state physics, nonlinear optics, stability of fluid motions, and dislocations in crystals where $\sin u$ is due to periodic structure of rows of atoms. The term $\sin u$ is the Josephson current across an insulator between two super conductors [1,2]. The sine-Gordon equation has applications to Josephson junctions, crystal dislocations, ultra-short optical pulses [17–27], relativistic field theory, and elementary particles [12,13]. Ultrashort pulses allow scientists to study ultrashort processes and allow for optical data transmissions [28–37], which may include electron dynamics within semiconductors, light-induced phase changes of metals, and chemical reactions.

In [13–15], an exact nonsingular solitary wave solution of the Schäfer-Wayne short pulse equation (SPE) is derived from the breather solution of the sine-Gordon equation by means of a transformation between these two integrable equations. The short pulse equation (SPE)

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{2}$$

related to the sine-Gordon equation, was derived by Schäfer and Wayne [13–15] as a model equation describing the propagation of ultra-short optical pulses in silica optical fibers. Moreover, the pulse solutions of the SPE were derived from the breather solutions of the sine-Gordon equation. The SPE is integrable by deriving its Lax pair [16,17]. Moreover, it was revealed that the sine-Gordon

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equation light bullets are similar to the Maxwell light bullets.

The aforementioned sine-Gordon equation (1) was examined thoroughly in the literature, numerically and analytically [1–17]. In [10], the chirp-free bright optical solitons to perturbed Fokas–Lenells equation were obtained via using two integration algorithms, namely the traveling wave hypothesis and the semi-inverse variational principle. Moreover, the adiabatic dynamics of the breather soliton of the sine-Gordon equation was examined thoroughly in [11] and some of the references therein. The sine-Gordon equation is completely integrable equation and gives multiple soliton solutions, and has important applications in solid state physics, nonlinear optics, optical solitons, and quantum field theory. The sine-Gordon equation was examined by using different numerical and analytical schemes. Most of the used algorithms depend mainly on working with the bilinear forms needed by the Hirota's bilinear method and the inverse scattering method.

Studies are flourishing on completely integrable equations because these equations are able to describe the real features of a variety of scientific applications. It is well known that completely integrable equations possess remarkable properties, such as infinitely many symmetries, infinitely many conservation laws, the Painlevé property, Bäcklund transformations, Darboux transformations, bilinear forms, Lax pair, Hamiltonian and bi-Hamiltonian structures, etc. A Painlevé integrable model indicates that this model possesses the Painlevé property.

It is worth noting that most of the extensive research work was focused on obtaining real soliton solutions, and mostly multiple real soliton solutions. In addition, researchers studied the structure of these soliton solutions and on the interactions of these soliton solutions related to shape, velocity and phase shifts. To the best of author's knowledge, much research has been done for the last decades. However, the complex solitons and the multiple complex soliton solutions have not been investigated in the literature. The primary purpose of the present paper is to introduce a new effective method which will allow us to construct complex soliton solutions, and in particular multiple complex soliton solutions for integrable equations. To gain further insight into the structures of the soliton solutions beyond real soliton features, we should recall that solitons can take on complex forms, such as dipole solitons, multi hump solitons, solitons organized as necklaces, and even complex beams carrying angular momentum, like rotating propellers [7,8]. In other words, integrable equations can also show an entirely complex soliton solutions, which will be examined in this work. The majority of research work on nonlinear equations was invested on real solutions, while complex solutions were examined on some equations such as Schrödinger equation and Ginzburg-Landau equation.

We mainly aim to show that the sine-Gordon equation gives multiple real soliton solutions and multiple complex soliton solutions as well. To achieve this goal, we will develop complex forms of the simplified Hirota's method and other methods to support our study. For comparison reasons, we will briefly report the multiple soliton solution for this equation. We then will introduce the complex forms of the simplified Hirota's method. We finally will close our work by employing the complex simplified Hirota's forms to formally derive multiple complex soliton solutions for the sine-Gordon equation (1).

2. Painlevé test

In this section we will use the Painlevé analysis to confirm the results obtained before in [1,2,9,10]. To show that the sine-Gordon equation is integrable, we introduce the transformation

$$v = e^{\mathrm{Iu}}, \quad I = \sqrt{-1}, \tag{3}$$

which leads to

$$u(x, t) = \frac{1}{I}\ln v(x, t), \quad I = \sqrt{-1}, \tag{4}$$

and

$$\sin u = \frac{1}{2I}\left(v - \frac{1}{v}\right). \tag{5}$$

Substituting (3)–(5) into (1) changes the sine-Gordon equation to an equivalent partial differential equation given as

$$2\mathrm{vv}_{\mathrm{xt}} - 2v_x v_t + v^3 - v = 0. \tag{6}$$

Assuming (6) has a solution as a Laurent expansion about a singular manifold $\psi = \psi(x, t)$ as

$$v(x, t) = \sum_{k=0}^{\infty} v_k(x, t)\psi^{k-\gamma}, \tag{7}$$

where $v_k(x, t)'s$ $(k = 0, 1, 2, \ldots)$ are the functions of x and t. On substitution of (7) in Eq. (6), then we can show that the characteristic equation for resonances has one branch with two resonances at $k = -1$ and 2. However, as usual, the resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x, t) = 0$. After detailed computations, we observed explicit expressions for u_1, and u_2 turns out to be arbitrary function and, hence compatibility condition, for $k = 2$, is satisfied identically which implies that equation (1) passes the Painlevé test for complete integrability. In what follows, we briefly present the multiple soliton solutions for the sine-Gordon equation obtained before by distinct methods in [1–14].