Optik - International Journal for Light and Electron Optics 172 (2018) 691–696

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.com/locate/ijleo

Original research article

Kinetic energy based relativistic wave equation

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ARTICLE INFO

Keywords:
Quantum mechanics
Special theory of relativity
Diffraction in time

ABSTRACT

The Schrödinger equation is based on a Hamiltonian, which is the sum of kinetic and potential energies. The energy term, in the relativistic energy-momentum relation, is divided into its kinetic and rest energy components according to the Hamiltonian of the Schrödinger equation. The differential equation is obtained by using the plane wave representation. It is found that the derived form of the Klein-Gordon equation is more similar to the Schrödinger equation than the classical one. Non-relativistic limit is also studied.

1. Introduction

The Klein-Gordon (KG) equation is a differential equation of quantum mechanics that is used for the explanation of the behaviors of spin-0 relativistic particles [1,2]. It was invented by Klein [3] and Gordon [4] in 1926. A natural tendency requires the reduction of the KG equation directly to the Schrödinger equation in the limit of non-relativistic speeds. However, an additional phase term, related with the rest energy of the particle, always exists in the wave function (ψ), in the non-relativistic case. In the probabilistic point of view, this situation does not cause any problem, since the probability density is evaluated from $\psi^*\psi$, which eliminates the phase function [5,6]. But there are quantum phenomena that take into account the effect of the wave function's phase term as the Aharonov-Bohm effect [7] and the spectral lines of the hydrogen atom [8]. This case is similar to the paraxial approximation of the wave equation in optics [9]. The solution of the paraxial equation must be multiplied by a plane wave function after the equation is solved. We used this approximation in order to transform the Schrödinger equation to a four dimensional Helmholtz equation [10]. This point puts forth that the KG equation does not rigorously reduce to the Schrödinger equation. The aim of this paper is to provide a new form for the KG equation that will not cause any additional phase function.

2. Theory

For the sake of completeness, we will investigate the reduction of the KG equation for the non-relativistic speeds. Our time convention is determined according to $\exp[-j(kx-\omega t)]$, which shows a plane wave propagating along the positive x direction. k and ω are the wave-number and angular frequency respectively. The relativistic equation can be given by

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{m_0^2c^2}{\hbar^2}\psi \tag{1}$$

for c is the speed of light. m_0 is the rest mass of the quantum particle and $\hbar$ angular Planck's constant. We will follow the procedure, given in [4]. It is assumed that the wave function has the expression

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https://doi.org/10.1016/j.ijleo.2018.07.074
Received 18 May 2018; Accepted 20 July 2018

$$\psi(\vec{r},t)=\varphi(\vec{r},t)e^{j\frac{m_0c^2}{\hbar}t} \tag{2}$$

according to our time convention. m_0c^2 is the rest mass of the particle. The non-relativistic approximation is determined by the limit $c\to\infty$ in the literature. The equation of

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-j\frac{2m_0}{\hbar}\frac{\partial\varphi}{\partial t}+\frac{m_0^2c^2}{\hbar^2}\varphi=\frac{m_0^2c^2}{\hbar^2}\varphi \tag{3}$$

is obtained when Eq. (2) is used in Eq. (1). Eq. (3) becomes

$$\nabla^2\varphi-j\frac{2m_0}{\hbar}\frac{\partial\varphi}{\partial t}=0 \tag{4}$$

which is the Schrödinger equation. Note that the actual solution of the differential equation is ψ, given in Eq. (2), not φ.

The aim of this study is the elimination of the middle step, given in Eq. (2). The relativistic energy-momentum relation can be expressed by

$$E^2=p^2c^2+m_0^2c^4 \tag{5}$$

where E and p are the energy and momentum respectively. The Hamiltonian can be given as

$$H=K+U \tag{6}$$

for the Schrödinger equation. K and U are the kinetic and potential energies. This means that only the kinetic and potential energies are taken into consideration in the development of the non-relativistic differential equation. We will use the same approach for the KG equation. The energy-momentum relation can be written as

$$(K+m_0c^2)^2=p^2c^2+m_0^2c^4 \tag{7}$$

in the relativistic case. K has the expression of

$$K=\frac{m_0c^2}{\beta}-m_0c^2 \tag{8}$$

for β is

$$\beta=\sqrt{1-\frac{v_g^2}{c^2}} \tag{9}$$

and v_g is the group velocity [11]. Eq. (7) can be arranged as

$$K^2+2m_0c^2K=p^2c^2. \tag{10}$$

The relations of

$$\frac{\partial}{\partial t}\equiv j\omega, \tag{11}$$

$$\frac{\partial^2}{\partial t^2}\equiv-\omega^2 \tag{12}$$

and

$$\nabla^2\equiv-k^2 \tag{13}$$

can be defined according to our time convention. Thus the new form of the KG equation can be written as

$$\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-j\frac{2m_0}{\hbar}\frac{\partial\psi}{\partial t}=0 \tag{14}$$

from Eq. (10). Note that the equations of

$$K=\hbar\omega \tag{15}$$

and

$$p=\hbar k \tag{16}$$

are also considered. Eq. (14) directly reduces to

$$\nabla^2\psi-j\frac{2m_0}{\hbar}\frac{\partial\psi}{\partial t}=0 \tag{17}$$

in the non-relativistic limit of $c\to\infty$. Eq. (17) is the exact expression of the Schrödinger equation of a free particle.