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Scattering of directed waves as an invertible Radon-to-Helmholtz mapping

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ABSTRACT

A concept of invertible Radon-to-Helmholtz mapping is proposed as a model of directed (i.e., paraxial) wave scattering. It is shown that not only the Mazar-Felsen (MF) solution, but also Born and Rytov approximations can be expressed in terms of a propagation operator that transforms (the complex exponential of) a linogram of the illuminated object into a set of its diffraction patterns. Since the propagation operator is easily invertible, a unified approach, based on either of these approximations, may be used to recover the scattering potential. For a purely refractive phantom, the superiority of the MF propagator as compared to the classical perturbative solutions is demonstrated.

1. Introduction

Elastic wave scattering is known to be a very powerful tool for a variety of imaging techniques, in particular, for either projective or tomographic imaging systems, operating in a very broad range of the electromagnetic spectrum, from microwaves, through visible light, to coherent X-ray radiation; see, e.g. [1–4]. For all of these applications, the scalar Helmholtz equation is typically used as the simplest model for connecting the measured wave field to the scattering potential of the object to be recovered. Unlike conventional tomographic imaging, which relies on the linear transformations of integral geometry, the full wavefield inversion problem is highly nonlinear. A number of perturbative solutions (e.g., the first-order Born and Rytov approximations), based on linearization of the Helmholtz operator, had been proposed in the past [5]. To get some insight into the physical meaning of these solutions, a spectral domain analysis is usually involved. It appears that the Born approximation, for example, describes the scattering process as an interaction of the incident wave with a corresponding resonant (Bragg) grating, hidden in the object function. This approach has been applied to evaluate the wave fields propagating in both deterministic and random media [5]; it has also been successfully used in the framework of linearized diffraction tomography (DT), to make the wavefield inversion [1,6].

Propagation, scattering, and diffraction of directed (a.k.a. forward scattered) waves can be efficiently described by a paraxial approximation of the Helmholtz equation. The resulting parabolic wave equation (PWE) may be solved perturbatively, and paraxial versions of both Born and Rytov approximations have long been in wide-spread use [5]. However, the list of available approximations (propagators) for PWE is much more diverse, including, in particular, a number of so-called straight-line solutions [7,8]. The most elaborated propagator of this type dates back to the paper by Mazar and Felsen (MF), published three decades ago [9].

Until recently, this propagator had not been actively used in wave scattering research. Only second- or fourth-order coherence functions, constructed in a similar fashion, have been studied to describe the propagation of directed waves in random media, e.g., laser beams in a turbulent atmosphere. The corresponding predictions have been shown to correlate reasonably well with both numerical simulations and experimental results, even in a regime of saturated scintillations, where the classical perturbative solutions diverge [7]. This fact may serve as an indirect indication of the potential of the MF propagator. Nevertheless, the validity and accuracy of the propagator itself have not yet been addressed. Also, it seems that the connection of this propagator (which is itself an approximation) to the classical perturbative solutions, either of the Born or Rytov type, has not been fully recognized.

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In the MF solution, the wave field is given as a weighted superposition of exponentials, each of which contains an integral over a straight line connecting the starting point in the source (input) plane, and the end point in the detector (output) plane. Hence, the physical interpretation of the results, and potentially, the inversion of the propagator, are more naturally to be performed in the configuration, not the momentum, space.

Integration over straight lines is immediately reminiscent of the computed tomography (CT) technique. In conventional tomography, the radiation is assumed to propagate along straight lines. Rotating the object, one can obtain a number of projections, jointly constituting a *sinogram*, i.e., a Radon space function that depends on the normal coordinates: the signed distance from the origin and the rotation angle [10]. The image of a point-like object is therefore a line in the form of the sine function in the Radon space, which explains the origin of the term. This is the commonly used parametrization of the straight line, but there are a number of alternatives. For one of them, proposed by Edholm and Herman [11], the line is defined by two points, one in the input plane, and the other in the output plane. The resulting *linogram* may also be considered an outcome of the Radon transformation, and, moreover, efficient algorithms exist to make the inversion based directly on a linogram dataset [12].

In this paper, we study the relationship between the Radon transform of the scattering potential (namely, the linogram), on the one hand, and the wave field satisfying the paraxial approximation of the Helmholtz equation (the corresponding diffraction pattern), on the other. Using a straight-path-integration approach, a Radon-to-Helmholtz (RtH) mapping is established from first principles, not only for the MF propagator (see Section 2), but also for the classical perturbative solutions, i.e., for the paraxial Born and Rytov approximations (Section 3). Since the RtH mapping is analytically invertible, all propagators considered here may serve as a starting point for an efficient configuration-space wavefield inversion. An example of such an inversion for a purely refractive phantom is presented (Section 4). It is shown that the MF propagator provides much better performance in recovering the object, as compared to the classical perturbative solutions. Possible generalizations of the proposed RtH mapping are also discussed (Section 5).

2. Straight-path integration

We start with the parabolic wave equation

$$2ik\partial_z g + \nabla_{\mathbf{r}}^2 g + k^2\varepsilon(\mathbf{r}, z)g(\mathbf{r}, z|\mathbf{r}_0, z_0) = 0, \tag{1}$$

supplemented by an initial condition of the form

$$g(\mathbf{r}, z_0|\mathbf{r}_0, z_0) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{2}$$

Function $\varepsilon(\mathbf{r}, z)$ is the scattering potential, i.e., a deviation of the permittivity from its background value, and $k = 2\pi/\lambda$ is the wave number. Hereafter, z is the longitudinal coordinate specifying the main propagation direction, and $\mathbf{r}$ denotes an m-dimensional ($m = 1$ or 2, respectively, for 2D or 3D formulations) position vector in the transverse plane. Although PWE is usually derived from the scalar Helmholtz equation, it has in fact a broader applicability. In particular, it describes equally well a forward scattering of electromagnetic waves (to a considerable extent, the polarization is conserved), and also may serve as an equation governing the wave function in transmission electron microscopy.

The solution of PWE in a homogeneous ($\varepsilon \equiv 0$) medium is given by

$$g_0(\mathbf{r}, z|\mathbf{r}_0, z_0) = \left[\frac{k}{2\pi\, i(z - z_0)}\right]^{m/2} \exp\left[\frac{ik(\mathbf{r} - \mathbf{r}_0)^2}{2(z - z_0)}\right]. \tag{3}$$

Although for an arbitrary inhomogeneous medium, an analytic solution of PWE does not exist, it can be presented symbolically in a Feynman path integral form,

$$g(\boldsymbol{r}, z|\boldsymbol{r}_0, z_0) = \int_{\boldsymbol{s}(z_0)=\boldsymbol{r}_0}^{\boldsymbol{s}(z)=\boldsymbol{r}} \mathrm{D}\boldsymbol{s}(\zeta)\exp\left\{ i\,\frac{k}{2}\int_{z_0}^{z} d\zeta\,[\,\dot{\mathbf{s}}^2(\zeta) + \varepsilon(\boldsymbol{s}(\zeta), \zeta)\,]\right\}, \tag{4}$$

where the integration $\mathrm{D}\boldsymbol{s}(\zeta)$ is interpreted as a sum of contributions of arbitrary trajectories, over which the wave propagates from point $\mathbf{r}_0$ in the input plane z_0 to point $\mathbf{r}$ in the output plane z [7,8]. Overdot is used in Eq. (4) to indicate the derivative with respect to the longitudinal pseudo-time coordinate ζ. The path integral is a very convenient form to start with, and to derive an (approximate) analytic solution. Usually, the latter is constructed by selecting only a restricted set of trajectories out of the continuum of all admissible paths $\mathbf{s}(\zeta)$. For example, the path can be expanded in a series of orthogonal functions, with an approximate solution that consists of only lower order harmonics [7].

Since our goal is to finally arrive at the conventional Radon transform, we have to confine ourselves to straight lines. The most complete set of straight lines is obtained when both start ($\mathbf{s}_0$) and end ($\mathbf{s}$) points of the path are allowed to be anywhere in the input and output planes, respectively. Although this formally contradicts the boundary conditions imposed on the path $\mathbf{s}(\zeta)$ in Eq. (4), it is, surprisingly, just this version (unlike, e.g., a set of appropriate sine functions which do respect the end-point restrictions) that leads to asymptotically exact results; see the corresponding analysis in Section 3. Further, the unknown propagator g is also parametrized by two points, $\mathbf{r}_0$ and $\mathbf{r}$. As a result, the infinite-dimensional path integral is likely to be reduced to a low- ($2m$ -) dimensional integral operator. Indeed, as follows from the analysis performed in [8], this strategy leads to a *nonlinear* RtH mapping,

$$h(\mathbf{r}_0, \mathbf{r}) = \mathscr{P}h_0(\mathbf{s}_0, \mathbf{s}) = \mathscr{P}\exp\left[iq_0(\mathbf{s}_0, \mathbf{s})\right], \tag{5}$$

where the propagation operator $\mathscr{P}$ transforms the so-called object hologram h_0 into a set of normalized diffraction patterns h. Formal