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Analytic methods for solving the cylindrical nonlinear Maxwell equations

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ABSTRACT

Cylindrical electromagnetic waves propagation in inhomogeneous and nonlinear media is of great interest for theory and application, and is an extremely complicated problem which can be described by the cylindrical nonlinear Maxwell equations. In this paper, we propose a method to exactly solve the cylindrical Maxwell equations in various types of inhomogeneous nonlinear media without dispersion. The obtained solutions can be used to describe the evolution of nonlinear oscillatory system and the interaction between two radiation waves in nonlinear media. We also show some nonlinear effects, such as second harmonic generation, and sum- and difference-frequency generation, result quite naturally from the solutions.

1. Introduction

The researches of cylindrical electromagnetic waves in inhomogeneous and nonlinear media are of fundamental importance [1,2], and can be applied to various engineering technologies, including electromagnetic fields amplification [3], controlling of photons [4], optical cloaking [5], geophysical prospecting [6] and others [7–11]. Due to the extreme complexity of the corresponding cylindrical nonlinear Maxwell equations [12], numerical methods [13–18] have long been the main means to conduct these researches. Despite this, however, exact solutions of the nonlinear system still play irreplaceable roles in understanding, predicting the nonlinear effects [19,20], and developing computational methods [21].

Recently, Ref. [2] proposed a method for constructing exact axisymmetric solutions of the cylindrical Maxwell equations which describe the behavior of cylindrical electromagnetic waves in a nonlinear nondispersive medium whose dielectric function is an exponential function of the electric field amplitude. Then, some related works have been reported [12,22–31]. For example, Refs. [23,24] showed this method could be extended to solve the problem of electromagnetic waves propagation in more complex media. Refs. [25,26] investigated second harmonic generation, and sum- and difference-frequency generation by using the exact solutions. However, to the best of our knowledge, there haven't been any other nonlinear dielectric functions be rigorously solved for identical physical models.

In this paper, we will construct exact solutions in two ways for the problem of cylindrical electromagnetic waves propagation in various types of inhomogeneous nonlinear and nondispersive media, and the physical meanings of the solutions are discussed.

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2. Electromagnetic model and exact solutions

We focus on the cylindrical Maxwell equations that have been used in some previous works [2,12,22–32]:

$$\frac{\partial H_\phi(r,t)}{\partial r} + \frac{H_\phi(r,t)}{r} = \frac{\partial D_z(r,t)}{\partial t} = \varepsilon(r,E_z)\frac{\partial E_z(r,t)}{\partial t},$$
$$\frac{\partial E_z(r,t)}{\partial r} = \mu_0 \frac{\partial H_\phi(r,t)}{\partial t},$$

which describe the features of electromagnetic fields in nonmagnetic and nondispersive media in a cylindrical coordinates system (r, ϕ, z), and the fields are independent of ϕ and z (consider E waves with respect to the z axis). Here E_z and H_ϕ represent, respectively, the electric and magnetic field, D_z is the electric displacement, $\varepsilon(r, E_z) = dD_z/dE_z$ is a dielectric function, and μ_0 is the permeability of vacuum. For convenience, we introduce the dimensionless variables $\rho = \frac{r}{r_0}$, $\tau = \frac{t}{r_0\sqrt{\epsilon_0\mu_0}}$ (ϵ_0 is the permittivity of vacuum), $E = \frac{E_z}{(\mathrm{N/C})}$, and $H = \frac{\sqrt{\mu_0}H_\phi}{\sqrt{\epsilon_0}(\mathrm{N/C})}$, where r_0 is a characteristic constant which describes the characteristic spatial scale, and (N/C) is the unit of electric field strength. Then we can get the dimensionless Maxwell equations as follows:

$$\begin{aligned}\frac{\partial H(\rho,\tau)}{\partial \rho} + \frac{H(\rho,\tau)}{\rho} &= \varepsilon(\rho,E)\frac{\partial E(\rho,\tau)}{\partial \tau},\\ \frac{\partial E(\rho,\tau)}{\partial \rho} &= \frac{\partial H(\rho,\tau)}{\partial \tau}.\end{aligned} \tag{1}$$

We will show that if the dielectric function ε is chosen in the form

$$\varepsilon(\rho,E) = \frac{f(\alpha E + \beta \ln \rho)}{\rho^2}, \tag{2}$$

where f is an arbitrary function and α and β ($\beta \neq 0$) are arbitrary constants, then Eq. (1) can be exactly solved. Dielectric function (2) indicates the nondispersive media here are inhomogeneous and nonlinear. When $\beta = 2$ and $f(x) = \exp(x)$, where x represents the independent variable, Eq. (2) will be reduced to $\varepsilon(\rho, E) = \exp(\alpha E)$. Such a dielectric function can be used to describe the dielectric properties of uniaxial pyroelectric and ferroelectric crystals [33–35], as is shown in Ref. [2]. With the development of material science, it is possible to construct almost any kind of media by considering metamaterials [36–38].

We use the following ansatz in Eqs. (1) and (2)

$$E = \alpha^{-1}(u - \beta\xi), \quad H = g_0^{-1}(v - \beta\eta), \tag{3}$$

where $g_0^{-1} = (\alpha\rho)^{-1}$, $\xi = \ln\rho$ and $\eta = \tau$, then we get

$$\begin{aligned}\frac{\partial v}{\partial \xi} &= f(u)\frac{\partial u}{\partial \eta},\\ \frac{\partial u}{\partial \xi} &= \frac{\partial v}{\partial \eta}.\end{aligned} \tag{4}$$

If the Jacobian $D(u, v)/D(\xi, \eta)$ of Eq. (4) is nonzero [2], then the hodograph transformation is applicable, and we view u and v as independent variables and ξ and η as dependent variables and obtain

$$\begin{aligned}\frac{\partial \eta}{\partial u} &= f(u)\frac{\partial \xi}{\partial v},\\ \frac{\partial \eta}{\partial v} &= \frac{\partial \xi}{\partial u}.\end{aligned} \tag{5}$$

Eq. (5) are linear equations. However, initial and boundary conditions of Eqs. (1) and (2) are hardly expressed by new variables u, v, ξ and η. Unlike the previous works [2,12,22–30], here we suppose that we have found an analytical solution of the following linear system:

$$\begin{aligned}\frac{\partial H(\rho,\tau)}{\partial \rho} + \frac{H(\rho,\tau)}{\rho} &= \frac{1}{\rho^2 f(F^{-1}(\ln\rho))}\frac{\partial E(\rho,\tau)}{\partial \tau},\\ \frac{\partial E(\rho,\tau)}{\partial \rho} &= \frac{\partial H(\rho,\tau)}{\partial \tau},\end{aligned} \tag{6}$$