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Novel analytical solutions of the fractional Drude model

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ABSTRACT

Alternative solutions for the equation of motion of the electrons in a metal in the presence of an external electric field are presented. Novel fractional derivatives of type Atangana–Koca–Caputo with constant and variable-order and fractional conformable derivative in the Liouville–Caputo sense were considered. The variable-order fractional derivative can be set as a smooth function, bounded on (0;1], while, the constant-order fractional derivative can be set as a fractional equation, bounded on (0;1]. We presented the exact solution using the properties of Laplace transform operator with its inversions. Numerical simulations are presented for evaluating the difference between these operators. Based on the generalized Mittag–Leffler function and power-law function with the conformable derivative, new behaviors for the optical properties in metals were obtained.

1. Introduction

Fractional calculus (FC) is a field becoming very popular of late due to its many potential applications. A dynamical process that modelled through fractional-order derivatives carries information about its present as well as past states [1–5]. Fractional differentiation involves integration over time from the past up to the present point of interest. FC has been considered as one of the best mathematic tools to characterize the memory property of complex systems and certain materials. These equations permits to model different physical processes with dissipation and long-range interaction [6–11]. Based on FC, several works in the context of electromagnetic theory are present in the literature, for example, the fractional curl operator and the fractional paradigm in electromagnetic theory was introduced in [12]. In [13], a generalized Helmholtz equation for wave propagation in fractional space was established and its analytical solution was obtained. In [14] the fractional Liouville–Caputo derivative was applied to study of the Drude model. Analytical solutions for the Drude model via Liouville–Caputo, Caputo–Fabrizio–Caputo and Atangana–Baleanu–Caputo fractional derivatives were obtained in [15]. The generalization of the complex electric conductivity and the complex dielectric function were reported in [16].

Based on the non-singular and non-local generalized stretched Mittag–Leffler function, Atangana and Baleanu proposed two fractional derivatives in the Liouville–Caputo and Riemann–Liouville sense. The fractional-order derivatives proposed by Atangana and Baleanu are at the same time filters and fractional operators [17]. In [18], Atangana and Koca presented a novel fractional-order derivative based on the generalized Mittag–Leffler function known as Prabhakar function.

Samko in [19] introduced the study of fractional integration and differentiation when the order is a function rather than a constant of arbitrary order. The variable-order fractional derivatives are very useful when investigating the memory properties which change with time and spatial location. Therefore, variable-order fractional derivative can be used to characterize

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variable memory effect of the system. In [20], the authors introduced a novel fractional variable-order derivative without singular kernel. This representation can be handled analytically and also have relationship with some integral transforms.

In [21], novel fractional derivatives in the Riemann–Liouville and Liouville–Caputo sense were proposed. These fractional derivatives were obtained iterating Riemann improper integrals. These new fractional conformable operators have interesting properties, for instance, depend on two fractional parameters naturally (α and β) which allows better detection of the memory of the physical systems. Interesting observation is when $\alpha = 1$ and $\beta \neq 1$, in this case, we recover the Riemann–Liouville or Liouville–Caputo fractional-order derivatives; when $\alpha \neq 1$ and $\beta = 1$, we recover the local conformable derivative proposed by Khalil's [22].

Based on the Atangana–Koca–Caputo fractional derivative with constant and variable-order, and the fractional conformable derivative in the Liouville–Caputo sense. We obtain alternative solutions for the equation of motion of the electrons in a metal in the presence of an external electric field. For the sake of comparison, numerical simulations were obtained for each derivative and compared graphically. The conductivity and the electrical resistivity, well as the thermal conductivity applying the Wiedemann–Franz law were presented. Novel behaviors for the optical properties described by the Drude model within the framework of the fractional calculus were presented. The manuscript is organized as follows: in Section 2, we recall the Mathematical background. In Section 3 alternative solutions for the Drude model are presented. Finally, Section 4 is devoted to our conclusions.

2. Basic definitions

We start with the mathematical background necessary to demonstrate the solution and modelling approaches. The fractional order derivatives of type Atangana–Koca–Caputo and the fractional conformable derivative in Liouville–Caputo sense are presented in this section.

Definition 1. The generalized Atangana–Koca–Caputo fractional derivative (AKC) with order ($\alpha > 0$) is defined as follows [18]

$$ {}^{0}_{AKC}D^{\alpha}_{t} f(t) = \frac{1}{g(\alpha)} \int_0^t \frac{d}{d\theta} f(\theta) \;\; E^{\alpha}_{\alpha,\alpha}[-g(\alpha)(t-\theta)^{\alpha}] \;\; d\theta. \tag{1} $$

The Laplace transform of Eq. (1) is defined as follows

$$ \mathcal{L}\{{}^{0}_{AKC}D^{\alpha}_{t} f(t)\}(s) = \frac{1}{g(\alpha)} \;\; (sF(s) - f(0)) \frac{s^{-n\alpha-1}}{(1-g(\alpha))^{\alpha}}. \tag{2} $$

Definition 2. For α, $\beta \in C$, with $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, the bi-parametric Mittag–Leffler function α, β, is defined by the series expansion as [24]

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{H_a} \frac{e^{\sigma}\sigma^{\alpha-\beta}}{\sigma^{\alpha} - z} d\sigma, \quad \alpha > 0, \;\; \beta > 0. \tag{3} $$

Considering $\alpha > 0$ and $\beta > 0$, the Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$ is given by

$$ \mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-1}, \quad \mathrm{Re}(s) > 1. \tag{4} $$

The mth derivative of the function $E^{(m)}_{\alpha,\beta}(at^{\alpha})$ with $\alpha > 0$ and $\beta > 0$ is given by

$$ E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \cdot \frac{a^k t^{\alpha k}}{\Gamma\left(\alpha k + \alpha m + \beta\right)}, \tag{5} $$

where $E^{(m)}_{\alpha,\beta}$ is the generalized Mittag–Leffler function.

The Laplace transform of the function $t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(at^{\alpha})$ is given as

$$ \mathcal{L}\left(t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right)\right)(s) = \frac{m!\,s^{\alpha-\beta}}{(s^{\alpha}-a)^{m+1}}, \quad \alpha > 0, \beta \in \mathbb{R}, \mathrm{Re}(s) > |a|^{1/\alpha}. \tag{6} $$

Definition 3. Let $f(x) \in C^1[a, b]$ and $g(x)$ a differentiable function in an open interval Ξ. The Atangana–Koca fractional variable-order derivative in Liouville–Caputo sense of $g(t)$ is given by [20]

$$ {}^{0}_{AKC}D^{f(x)}_{t} g(t) = \int_0^t \frac{dg(s)}{ds} \quad \exp[-f(x)(t-s)] \;\; ds, \tag{7} $$

where ${}^{0}_{AKC}D^{f(x)}_{t} g(t)$ is the Atangana–Koca–Caputo fractional variable-order derivative. This operator is a particular case of Eq. (1) with variable-order $f(x)$.