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# Conditions required to make gauge-invariant the electromagnetic spin and orbital angular momenta and helicity densities in isotropic and anisotropic media

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ABSTRACT

The spin and orbital angular momenta densities ($\boldsymbol{s}_n$ and $\boldsymbol{l}_n$) and helicity density ($h_n$) of the electro- magnetic (EM) fields are defined using an electric vector potential $\boldsymbol{C}$ in analogy with the magnetic vector potential $\boldsymbol{A}$, where $\boldsymbol{C}$ is introduced to observe the duality invariance and electric-magnetic democracy. However, their definitions for $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ have an essential drawback (lacking for the gauge invariance) that they vary widely depending on the choice of two kinds of $\boldsymbol{A}$ and $\boldsymbol{C}$. To solve this problem, it is necessary to make $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ gauge-invariant. The gauge invariance for $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ in isotropic and anisotropic media can be established by an appropriate combination of gauge functions and scalar potentials. As a result, it is proved in a distinct way from the second quantization procedure that the gauge invariance for $\boldsymbol{s}_n$ and $\boldsymbol{l}_n$ depending on the phase difference between $\boldsymbol{A}$ and $\boldsymbol{C}$ leads to the quantization of electromagnetic spin. Moreover, the gauge-invariant $h_n$ gives a reasonable explanation for the Faraday rotation caused in an anisotropic medium.


## 1. Introduction

The angular momentum (AM) carried by light can be characterized by the spin angular momentum (SAM) associated with the circular polarization (CP) and orbital angular momentum (OAM) associated with the spatial distribution of the electromagnetic (EM) wave. In practice, both SAM and OAM of a light beam have been measured [1–3]. Theoretically, the old SAM and OAM densities have been defined as $\boldsymbol{s}_o = (\boldsymbol{D} \times \boldsymbol{A})$ and $\boldsymbol{l}_o = \Sigma_{i=x,\,y}\, D_i\, (\boldsymbol{r} \times \nabla) A_i$ for the transverse plane EM waves propagating along the $z$ axis, where the symbol $\nabla$ is the gradient operator, $\boldsymbol{D}$ is the electric induction field and $\boldsymbol{A}$ is the magnetic vector potential defined as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{B}$ is the magnetic induction field [4–7].The SAM arises in the transverse plane EM waves, while the OAM never appears there. When the phase gradient satisfies certain conditions providing the transverse energy circulation within the beam, however, the OAM arises sometimes [8,9].On the other hand, the old helicity density $h_o$ has been defined as $h_o = \boldsymbol{H} \cdot \boldsymbol{A}$ for the transverse plane EM waves traveling along the $z$ axis, where $\boldsymbol{H}$ is the magnetic field [10].

The old SAM density $\boldsymbol{s}_o = (\boldsymbol{D} \times \boldsymbol{A})$ defined above for the transverse plane EM waves traveling along the $z$ axis does not contain the $\boldsymbol{D}$ and $\boldsymbol{B}$ fields equivalently, resulting in the breakdown of the electric-magnetic democracy [5]. The same applies to the helicity density $h_o = \boldsymbol{H} \cdot \boldsymbol{A}$, because $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are not included equivalently in $h_o$. For this reason, the SAM ($\boldsymbol{s}_n$), OAM ($\boldsymbol{l}_n$) and helicity ($h_n$) densities are defined here as $\boldsymbol{s}_n = [(\boldsymbol{D} \times \boldsymbol{A}) + (\boldsymbol{C} \times \boldsymbol{B})]/2$, $\boldsymbol{l}_n = \Sigma_{j=x,y,z}\, [D_j (\boldsymbol{r} \times \nabla) A_j - B_j (\boldsymbol{r} \times \nabla) C_j]/2$ and $h_n = (\boldsymbol{A} \cdot \boldsymbol{H} + \boldsymbol{C} \cdot \boldsymbol{E})/2$ for the optical fields, to leave invariant under the duality transformations of $(\boldsymbol{E}, \boldsymbol{H}) \leftrightarrow (\boldsymbol{H}, -)$, $(\boldsymbol{D}, \boldsymbol{B}) \leftrightarrow (\boldsymbol{B}, -)$ and $(\boldsymbol{A}, \boldsymbol{C}) \leftrightarrow (-\boldsymbol{C}, \boldsymbol{A})$ which result from the interchange of the electricity and magnetism [11], where $\boldsymbol{E}$ is the electric field and $\boldsymbol{D}$ is defined as $\boldsymbol{D} = \nabla \times \boldsymbol{C}$ using an electric vector potential $\boldsymbol{C}$. There are also similar definitions for the spin density $\boldsymbol{s}_d$

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and helicity density $h_d$ in the dual electromagnetism [12–14]. However, $\boldsymbol{s}_d$ and $h_d$ are inapplicable to anisotropic media, because they originate basically from the free-field Maxwell's equations. It will be shown here that $h_n$ defined here gives a reasonable explanation for the Faraday rotation caused in an optically active medium [15].

As is clear from the total angular momentum density $\boldsymbol{j}$ defined as $\boldsymbol{j} = \boldsymbol{r} \times (\boldsymbol{D} \times \boldsymbol{B})$ for the optical fields [13], $\boldsymbol{j}$ is invariant under the gauge transformation, but both $\boldsymbol{s}_o$ and $\boldsymbol{l}_o$ generated [5,6,14] by the partial integration of $\boldsymbol{j}$ due to the introduction of the vector potential $\boldsymbol{A}$ lack generally a gauge invariance. Even if $\boldsymbol{s}_o$ and $\boldsymbol{l}_o$ are improved to have both duality invariance and electric-magnetic democracy, their gauge invariances are not recovered at all. This is because the physical quantities of $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ which are defined above vary significantly depending on the choice of two kinds of $\boldsymbol{A}$ and $\boldsymbol{C}$. In general, the observable physical quantity must be invariant under the gauge transformation. For this reason, it is investigated here whether the gauge invariances for $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ can be established by an appropriate combination of gauge functions and scalar potentials contributing to $\boldsymbol{A}$ and $\boldsymbol{C}$ and what conditions are required to acquire the gauge invariance in isotropic and anisotropic media.

The first purpose of this study is to find the conditions required to make $\boldsymbol{s}_n$, $\boldsymbol{l}_n$ and $h_n$ gauge-invariant, and the second one is to show that $h_n$ gives a reasonable explanation for the Faraday rotation occurring in an anisotropic (opticall active) medium.

## 2. Analysis

### 2.1. Maxwell's equations

The Maxwell equations for the EM fields in the absence of free charges or electric currents are given in SI units by [16]

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1}$$

$$\nabla \times \boldsymbol{H} = -\frac{\partial \boldsymbol{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3}$$

$$\nabla \cdot \boldsymbol{D} = 0 \tag{4}$$

where the electric $\boldsymbol{D}$ and magnetic $\boldsymbol{B}$ inductions are related to the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ through $\boldsymbol{D} = \varepsilon\, \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, where $\varepsilon$ and $\mu$ represent the permittivity and permeability tensors, respectively. Maxwell's equations in the absence of free charges or electric currents are invariant by putting $(\boldsymbol{E}, \boldsymbol{H}) \to (\boldsymbol{H}, -\boldsymbol{E})$ and $(\boldsymbol{D}, \boldsymbol{B}) \to (\boldsymbol{B}, -\boldsymbol{D})$ in the optical fields. This duality transformation leaves quadratic forms such as $(\boldsymbol{E} \times \boldsymbol{H})$, $(\boldsymbol{E}{\cdot}\boldsymbol{D} + \boldsymbol{H}{\cdot}\boldsymbol{B})$ and the components of the Maxwell stress tensor $T_{ij}$ invariant, where $i$, $j = x$, $y$ and $z$ [16]. The Maxwell's equations have an important feature that they are invariant under the duality and gauge transformations.

### 2.2. Old spin and orbital angular momenta in an isotropic medium

The total angular momentum $\boldsymbol{J}$ of the EM fields has been defined as [16]

$$\boldsymbol{J} = \int_\tau [\boldsymbol{r} \times (\boldsymbol{D} \times \boldsymbol{B})]\ \mathrm{d}v = \int_\tau \boldsymbol{j}\ dv \tag{5}$$

in analogy to the interpretation of the linear momentum density $(\boldsymbol{D} \times \boldsymbol{B})$, where $\boldsymbol{j}$ is the angular momentum density and is gauge-invariant. Here a medium is assumed to be loss-free and homogeneous. This definition is invariant under the duality transformation and is valid at least when the medium is linear but not necessarily isotropic in its response. The magnetic induction field $\boldsymbol{B}$ is expressed by the magnetic vector potential $\boldsymbol{A}$ as [16]

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{6}$$

By substituting Eq. (6) into Eq. (5), the old SAM ($\boldsymbol{S}_o$) and OAM ($\boldsymbol{L}_o$) for EM waves traveling along the $z$ direction in an isotropic medium have been described as [4–7]

$$\boldsymbol{S}_o = \int_\tau dv s_o = \int_\tau dv\,(\boldsymbol{D} \times \boldsymbol{A}) \tag{7}$$

and

$$\boldsymbol{L}_o = \int_\tau \mathrm{d}v\ \ \boldsymbol{l}_o = \int_\tau \mathrm{d}v\, \Sigma_{j=x,y,z} D_j (\boldsymbol{r} \times \nabla) A_j, \tag{8}$$

where $\tau$ is a volume integrated over the wave packet along the trajectory. As is clear from Eqs.(7) and (8), the old SAM and OAM densities ($\boldsymbol{s}_o$ and $\boldsymbol{l}_o$) do not contain equivalently $\boldsymbol{D}$ and $\boldsymbol{B}$ fields, resulting in the breakdown of the electric-magnetic