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From the classical diffusion equation to quantum master equation for describing optical diffusion[☆]

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ABSTRACT

We study quantum-mechanical diffusion master equation by comparison with the classical diffusion equation, and deduce how a pure coherent state evolves into a mixed state by diffusion both in classical way and in quantum mechanical way.

1. Introduction

In optical physics [1], the diffusion equation describes the behavior of the collective motion of photons in a space resulting from the random movement of each photon. The classical linear diffusion equation is [2]

$$\frac{\partial P(t)}{\partial t} = \kappa \nabla^2 P(t), \tag{1}$$

where $P(t)$ is distribution function, κ is the diffusion constant. In 2-dimensional case

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{2}$$

Let $z = re^{i\varphi}$, then we can derive

$$\frac{\partial}{\partial z} = \frac{1}{2}e^{-i\varphi}\left(\frac{\partial}{\partial r} + \frac{1}{ir}\frac{\partial}{\partial \varphi}\right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2}e^{i\varphi}\left(\frac{\partial}{\partial r} - \frac{1}{ir}\frac{\partial}{\partial \varphi}\right), \tag{3}$$

thus

$$\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) = \frac{\partial^2}{\partial z^* \partial z}, \tag{4}$$

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and the classical linear diffusion Eq. (1) can be recast in the form

$$\frac{\partial P(z,t)}{\partial t}=\kappa\frac{\partial^2 P(z,t)}{\partial z\partial z^*}. \tag{5}$$

In this work we shall stem from Eq. (5) and the coherent state representation to derive the quantum master equation for describing optical diffusion which is [3]

$$\frac{d\rho(t)}{dt}=-\kappa(a^\dagger a\rho-a^\dagger\rho a-a\rho a^\dagger+\rho aa^\dagger), \tag{6}$$

where ρ is density operator of some optical field, $a^\dagger$, a are photon creation and annihilation operators, obeying

$$\left[a,a^\dagger\right]=1, \tag{7}$$

and then explore its solution comparing with the classical diffusion equation's solution. The motivation of this work is to answer many people's wondering why Eq. (6) denotes quantum mechanical diffusion process, does it really corresponds to classical diffusion? In the following we report an elegant and concise way to connect quantum diffusion to classical diffusion.

2. How come from (5) to (6)

The coherent state is defined as [4,5]

$$|z\rangle=e^{-|z|^2/2+za^\dagger}|0\rangle, \tag{8}$$

which is eigenstate of the annihilation operator

$$a|z\rangle=z|z\rangle \tag{9}$$

and composes the completeness relation

$$\int\frac{d^2z}{\pi}|z\rangle\langle z|=1. \tag{10}$$

Using the method of integration within ordered product (IWOP) of operators and the normally ordered form of vacuum state [6]

$$|0\rangle\langle 0|=:e^{-a^\dagger a}:, \tag{11}$$

we can put $|z\rangle\langle z|$ into [7]

$$|z\rangle\langle z|=:e^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:, \tag{12}$$

and derive

$$\int\frac{d^2z}{\pi}|z\rangle\langle z|=\int\frac{d^2z}{\pi}:e^{-|z|^2+za^\dagger+z^*a-a^\dagger a}:=1. \tag{13}$$

Any density operator $\rho(t)$ has its so-called P-representation onto the coherent state basis

$$\rho(t)=\int\frac{d^2z}{\pi}P(z,t)|z\rangle\langle z|, \tag{14}$$

where $P(z,t)$ is a classical function. Using

$$\langle z'|z\rangle=e^{-|z|^2/2-|z'|^2/2+z'^*z} \tag{15}$$

we know

$$\begin{aligned}\langle -z'|\rho|z'\rangle&=\langle -z'|\int\frac{d^2z}{\pi}P(z,t)|z\rangle\langle z|z'\rangle\\&=e^{-|z'|^2}\int\frac{d^2z}{\pi}P(z)e^{-|z|^2-z'^*z+z^*z'},\end{aligned} \tag{16}$$

since $(z^*z'-z'^*z)$ is pure imaginary we can consider this equation as the Fourier transformation of $P(z)e^{-|z|^2}$ then its inverse transformation is

$$P(z)e^{-|z|^2}=\int\frac{d^2z'}{\pi}\langle -z'|\rho|z'\rangle e^{|z'|^2+z'^*z-z^*z'}. \tag{17}$$