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Modeling the fractional non-linear Schrödinger equation via Liouville-Caputo fractional derivative

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ABSTRACT

In this paper the modified homotopy analysis transform method is applied to obtain approximate analytical solutions of the time-fractional non-linear Schrödinger equation. The fractional derivative is described in the Liouville-Caputo sense. The solutions are obtained in the form of rapidly convergent infinite series with easily computable terms. New exact solutions are constructed under constraint conditions. Employing theoretical parameters, we present some numerical simulations.

1. Introduction

The non-linear Schrödinger equations (NLSE) describing wave propagation in dispersive and inhomogeneous media, such as plasma phenomena, nonlinear optics, hydrodynamics, non-uniform dielectric media and the propagation of optical pulse in nonlinear media [1–12]. This equation exhibit solitary type solutions which are often applied to optical pulses in optical fibers, ultra-short optical solitons propagate in nonlinear medium and optical fibers.

Fractional Calculus (FC) represents complex physical phenomena more accurate and efficient than classical calculus. In recent years, many field of sciences and technology have used fractional order derivatives to model many real world problems. The fractional orders of differentiation highlight the intermediate behaviours that cannot be modeled by ordinary equations. FC shows superiorities in describing complex dynamical systems associated with system memory and hereditary properties of various processes [13–17]. In general, the integer model does not carry any information about the memory and learning mechanism. In the evolution equations, the growth rates depend only on the current state. Nevertheless, in many situations the growth rate is also dependent on the history of that density or its memory. The fractional-order differential equations have non-local properties (this means that the next state of a model depends not only upon its current state, but also upon all its preceding states, which expresses that the system's response at any time will be affected by all previous

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responses), so consider the past and distributed effects of any physical system. Therefore, any dynamical process modeled through fractional order differential equations has a memory effect [18].

Exists in the literature several definitions of fractional-order derivatives can be used to solve the problems of fractional differential equations. These definitions include, Riemann-Liouville, Grünwald-Letnikov, Riesz, Liouville-Caputo, among others [19]. The Liouville-Caputo fractional-order derivative is based on the convolution of the local derivative of a given function with power law function. For this operator, the derivative of a constant is zero and we can define, properly, the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical ordinary derivative.

Finding accurate and efficient methods for solving fractional non-linear differential equations (FNLDEs) has been an active research undertaking. Many methods are usually used to handle the non-linear equations, for instance, Hermite collocation method [20], invariant subspace method [21], optimal homotopy asymptotic method [22], q-homotopy analysis transform technique [23], Adomian descomposition methods [24] and homotopy analysis Laplace transform method [25,26]. The homotopy analysis transform method with homotopy polynomials (MHATM) was proposed in [27], this method is an analytical technique based in the combination of the homotopy analysis method and Laplace transform with homotopy polynomial. In [27] the MHATM was considered to solving the time-fractional Keller-Segel model considering the Liouville-Caputo fractional derivative. A convergence analysis of MHATM was obtained by the proposed method and verified through different graphical representations.

The aim of this paper is obtain analytical and approximate solutions for fractional NLSE which describes the propagation of ultrashort optical solitons in high-order dispersion using homotopy analysis transform method with homotopy polynomials via Liouville-Caputo fractional-order derivatives. The paper is organized as follows. The generalized nonlinear Schrödinger equation and the MHATM are discussed in the Section 2. The application of the MHATM and numerical simulations are reported in Sections 3 and 4, the conclusions are discussed.

2. Generalized non-linear fractional Schrödinger equation

The generalized non-linear Schrödinger equation describes the propagation of ultra-short optical solitons through parabolic law medium in high-order dispersion [1]. In Liouville-Caputo sense we have the following equation

$$
\begin{aligned}
&{}_0^C\mathcal{D}_t^{4\alpha}u(x,t) + a\,{}_0^C\mathcal{D}_t^{3\alpha}u(x,t) + b\,{}_0^C\mathcal{D}_t^{2\alpha}u(x,t) \\
&= \frac{24}{\beta_4}\left(i\frac{\partial u}{\partial x} + \gamma_1|u|^2u + \gamma_2|u|^4u + is\left(|u|^2\frac{\partial u}{\partial t} + u\frac{\partial |u|^2}{\partial t}\right)\right),
\end{aligned}
\tag{1}
$$

with initial conditions

$$
u(x,0) = e^{ix}, \quad u'(x,0) = u''(x,0) = u'''(x,0) = 0, \tag{2}
$$

where $a = i\frac{24\beta_3}{\beta_4}$ and $b = \frac{12\beta_2}{\beta_4}$.

In Eq. (1), the depend variable $u(x, t)$ represent the normalized electric-field envelope and ${}_0^C\mathcal{D}_t^{\alpha}$ are Liouville-Caputo fractional-order derivatives. This equation consider the group velocity dispersion, third-order dispersion, fourth-order dispersion and self-steepening [1]. The variable x represents the longitudinal coordinate along the optical fibers and t represents the time coordinate in a reference frame that moves with the pulse group velocity [28].

The Liouville-Caputo fractional derivative (C) is presented as [19]

$$
{}_0^C\mathcal{D}_t^{\alpha}\{f(t)\} = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{d}{dt}f(\tau)(t-\tau)^{-\alpha}d\tau, \quad n-1 < \alpha \le n, \tag{3}
$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Laplace transform to Liouville-Caputo fractional-order derivative gives [19]

$$
\mathcal{L}\left\{{}_0^C\mathcal{D}_t^{\alpha}\{f(t)\}\right\}(s) = S^{\alpha}F(S) - \sum_{k=0}^{m-1}S^{\alpha-k-1}f^{(k)}(0). \tag{4}
$$

3. Homotopy analysis transform method with homotopy polynomials

The modified homotopy analysis transform method (MHATM) was proposed in [27]. The method is an analytical technique based in the combination of the homotopy analysis method and Laplace transform with homotopy polynomial. The main steps of this method are described as follows:

Step 1. Let us consider the following equation

$$
\mathcal{D}_t^{\alpha}\{f(x,t)\} + \Xi[x]f(x,t) + \Lambda[x]f(x,t) = \Psi(x,t), \quad t > 0, \quad x \in \Re, \quad 0 < \alpha \le 1, \tag{5}
$$