Frequency domain conditions for the existence of Bohr almost periodic solutions in evolution equations

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Abstract: We consider a control problem for the heating process of an elastic plate. The heat flux within the plate is modeled by the heat equation with nonlinear Neumann boundary conditions according to Newton's law. As input at a part of the boundary we take the nonlinearly transformed and modulated heat production of a separate heater which is given by a nonlinear Duffing-type ODE. This ODE depends on measurements of the temperature within the plate and on Bohr resp. Stepanov almost periodic in time forcing terms. The physical problem is generalized to a bifurcation problem for non-autonomous evolution systems in rigged Hilbert spaces. Using Lyapunov functionals, invariant cones and monotonicity properties of the nonlinearities in certain Sobolev spaces, we derive frequency domain conditions for the existence and uniqueness of an asymptotically stable and almost periodic in time temperature field.

Keywords: Periodic motion, stability analysis, partial differential equations, frequency domains, control closed-loop

1. INTRODUCTION

Let us introduce some function spaces. We follow the representation in Pankov [1990]. Suppose $(E, \|\cdot\|_E)$ is a Banach space.

If $J \subset \mathbb{R}$ is an interval, denote by $C(J;E)$ the space of all continuous functions from J to E, endowed with the topology of uniform convergence on compact sets. If $J = \mathbb{R}$ or $J = \mathbb{R}_+$ the space $C_b(J;E)$ is the subspace of $C(J;E)$ of bounded functions equipped with the norm

$$\|f\|_{C_b} := \sup_{u\in J} \|f(u)\|_E \,.$$

The Banach space of *Stepanov bounded* on $J = \mathbb{R}$ or $J = \mathbb{R}_+$ *functions* (of exponent $p = 2$) is the space $BS^2(J;E)$ which consists of all functions $f \in L^2_{\text{loc}}(J;E)$ having finite norm

$$\|f\|^2_{S^2} := \sup_{t\in J} \int_t^{t+1} \|f(\tau)\|^2_E \, d\tau \,.$$

A subset $\mathcal{S} \subset \mathbb{R}$ is *relatively dense* if there is a compact interval $\mathcal{K} \subset \mathbb{R}$ such that $(s + \mathcal{K}) \cap \mathcal{S} \neq \varnothing$ for all $s \in \mathbb{R}$. A function $f \in C_b(\mathbb{R}\,;E)$ is said to be *Bohr almost periodic* if for any $\varepsilon > 0$ the set

$$\{\tau \in \mathbb{R} \,|\, \sup_{s\in\mathbb{R}} \|f(s+\tau) - f(s)\| \le \varepsilon\}$$

of *ε-almost periods* is relatively dense in $\mathbb{R}$.

For a function $f \in L^2_{\text{loc}}(\mathbb{R};E)$, put

$$f^b(t) := f(t+w), \; w \in [0,1], t \in \mathbb{R} \,.$$

The function $f^b(t)$ is regarded as a function with values in the space $L^2(0,1;E)$. Then

$$BS^2(\mathbb{R};E) = \{f \in L^2_{\text{loc}}(\mathbb{R};E) | f^b \in L^\infty(\mathbb{R};L^2(0,1;E))\}$$

and, moreover, $\|f\|_{S^2} = \|f^b\|_{L^\infty}$.

A function $f \in BS^2(\mathbb{R};E)$ is called an *almost periodic function in the sense of Stepanov and of exponent* 2 (abbreviated S^2-a.p.) if $f^b \in \text{CAP}(\mathbb{R};L^2(0,1;E))$. In this case the ε-almost periods of f^b are called the ε-almost periods of f. The space of S^2-a.p. functions with values in E is denoted by $S^2(\mathbb{R};E)$. Obviously, CAP $(\mathbb{R};E) \subset S^2(\mathbb{R};E)$.

2. CONTROL SYSTEMS IN LUR'E FORM WITH A DUFFING TYPE NONLINEARITY

Let $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ be a Gelfand rigging of the real Hilbert space $\mathcal{V}_0$, i.e. a chain of Hilbert spaces with dense and continuous inclusions. Denote by $(\cdot,\cdot)_{\mathcal{V}_j}$ and $\|\cdot\|_{\mathcal{V}_j}, j = 1, 0, -1$, the scalar product resp. norm in $\mathcal{V}_j (j = 1, 0, -1)$ and by $(\cdot,\cdot)_{\mathcal{V}_{-1},\mathcal{V}_1}$ the pairing between $\mathcal{V}_{-1}$ and $\mathcal{V}_1$.

Let $A_0 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$ be a linear operator, $b_0 \in \mathcal{V}_{-1}$ a generalized vector, $c_0 \in \mathcal{V}_0$ a vector and $d_0 < 0$ a number. According to the vectors c_0 and b_0 we introduce the linear operators $C_0 \in \mathcal{L}(\mathcal{V}_0, \mathbb{R})$ and $B_0 \in \mathcal{L}(\mathbb{R}, \mathcal{V}_{-1})$ by $C_0\nu = (c_0, \nu)_{\mathcal{V}_0}$, $\forall\, \nu \in \mathcal{V}_0$, and $B_0\xi := \xi b_0$, $\forall \xi \in \mathbb{R}$.

Assume that $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two scalar-valued functions. Our aim is to study a system of indirect control Leonov et al. [1992], which is formally given as

$$\begin{aligned} \dot\nu &= A_0\nu + b_0[\phi(t,z) + g(t)]\,, \\ \dot z &= (c_0,\nu)_{\mathcal{V}_0} + d_0[\phi(t,z) + g(t)]\,. \end{aligned} \qquad (1)$$

Let us demonstrate how (1) can be written as a standard control system. Consider for this the Gelfand rigging $Y_1 \subset Y_0 \subset Y_{-1}$, in which

$$Y_j := \mathcal{V}_j \times \mathbb{R}\,, \quad j = 1, 0, -1\,. \qquad (2)$$

The scalar product $(\cdot,\cdot)_j$ in Y_j is introduced as

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$\big((\nu_1,z_1),(\nu_2,z_2)\big)_j := (\nu_1,\nu_2)_{\mathcal{V}_j} + z_1 z_2$, where $(\nu_1,z_1),(\nu_2,z_2) \in Y_j$ are arbitrary. The pairing between Y_{-1} and Y_1 is defined for $(h,\xi) \in \mathcal{V}_{-1}\times\mathbb{R} = Y_{-1}$ and $(\nu,\varsigma) \in \mathcal{V}_1\times\mathbb{R} = Y_1$ through

$$((h,\xi),(\nu,\varsigma))_{-1,1} := (h,\nu)_{\mathcal{V}_{-1},\mathcal{V}_1} + \xi\,\varsigma\,. \tag{3}$$

Let $b := \begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \in Y_{-1}$ and $c := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in Y_0$. Suppose further that the operators $C \in \mathcal{L}(Y_0,\mathbb{R})$ and $B \in \mathcal{L}(\mathbb{R},Y_{-1})$ are given as

$$Cy = (c,y)_0\,, \quad \forall\, y \in Y_0\,, \qquad B\xi = \xi b\,, \quad \forall\, \xi \in \mathbb{R}\,,$$

and the operator $A \in \mathcal{L}(Y_1,Y_{-1})$ is defined as

$$A := \begin{bmatrix} A_0 & 0 \\ C_0 & 0 \end{bmatrix}.$$

Consider now the system

$$\dot y = Ay + B\,[\phi(t,z) + g(t)]\,,\ z = Cy\,, \tag{4}$$

which is equivalent to (1) through $y = (\nu,z)$. If $-\infty \le T_1 < T_2 \le +\infty$ are arbitrary, we define the norm for Bochner measurable functions in $L^2(T_1,T_2;Y_j)$, $j = 1,0,-1$, by

$$\|y\|_{2,j} := \left(\int_{T_1}^{T_2} \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{5}$$

Let $\mathcal{W}(T_1,T_2;Y_1,Y_{-1})$ be the space of functions y such that $y \in L^2(T_1,T_2;Y_1)$ and $\dot y \in L^2(T_1,T_2;Y_{-1})$, equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2;Y_1,Y_{-1})} := \left(\|y\|_{2,-1}^2 + \|\dot y\|_{2,-1}^2\right)^{1/2}. \tag{6}$$

Let us introduce the following assumptions **(A1)** – **(A6)** about the operator $A_0 \in \mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$, the vectors $b_0 \in \mathcal{V}_{-1}$ and $c_0 \in \mathcal{V}_0$, and the functions ϕ and g.

(A1) For any $T > 0$ and any $(f_1,f_2) \in L^2(0,T;\mathcal{V}_{-1}\times\mathbb{R})$ the problem

$$\begin{aligned} &\dot\nu = A_0\nu + f_1(t)\,, \\ &\dot z = (c_0,\nu)_{\mathcal{V}_0} + f_2(t)\,, \quad (\nu(0),z(0)) = (\nu_0,z_0) \end{aligned} \tag{7}$$

is well-posed, i.e. for arbitrary $(\nu_0,z_0) \in Y_0$, $(f_1,f_2) \in L^2(0,T;\mathcal{V}_{-1}\times\mathbb{R})$ there exists a unique solution $(\nu,z) \in \mathcal{W}(0,T;Y_1,Y_{-1})$ satisfying (7) in a variational sense and depending continuously on the initial data, i.e.

$$\begin{aligned} &\|(\nu,z)\|_{\mathcal{W}(0,T;Y_1,Y_{-1})}^2 \ \le \\ &k_1\|(\nu_0,z_0)\|_{\mathcal{V}_0\times\mathbb{R}}^2 \ + \ k_2\|(f_1,f_2)\|_{2,-1}^2\,, \end{aligned} \tag{8}$$

where $k_1 > 0$ and $k_2 > 0$ are some constants .

(A2) There is a $\lambda > 0$ such that $A_0 + \lambda I$ is a Hurwitz operator .

(A3) For any $T > 0, (\nu_0,z_0) \in \mathcal{V}_1\times\mathbb{R}, (\tilde\nu_0,\tilde z_0) \in \mathcal{V}_1\times\mathbb{R}$ and $(f_1,f_2) \in L^2(0,T;\mathcal{V}_1\times\mathbb{R})$ the solution of the direct problem (7) and the solution of the adjoint problem

$$\begin{aligned} \dot{\tilde\nu} &= -(A_0^+ + \lambda\,I)\tilde\nu + f_1(t) \\ \dot{\tilde z} &= -C_0^+\,\tilde z - \lambda\,\tilde z + f_2(t) \end{aligned} \tag{9}$$

are strongly continuous in t in the norm of $\mathcal{V}_1\times\mathbb{R}$.

(A4) The pair (A_0,b_0) is L^2-controllable, i.e. for arbitrary $\nu_0 \in \mathcal{V}_0$ there exists a control $\xi\,(\cdot) \in L^2(0,\infty;\mathbb{R})$ such that the problem

$$\dot\nu = A_0\nu + b_0\xi\,, \quad \nu(0) = \nu_0$$

is well-posed in the variational sense on $(0,\infty)$.

Introduce by (c denotes the complexification)

$$\chi(p) = \left(c_0^c, (A_0^c - pI^c)^{-1}\,b_0^c\right)_{\mathcal{V}_0}\,, \quad p \in \rho(A_0^c)$$

the transfer function of the triple (A_0^c, b_0^c, c_0^c) .

(A5) Suppose $\lambda > 0$ and $\kappa_1 > 0$ are parameters, where λ is from **(A2)**. Then:

$$\begin{aligned} &a)\ \lambda d_0 + \mathrm{Re}\,(-i\omega - \lambda)\chi(i\omega - \lambda) + \\ &\kappa_1\,|\,\chi(i\omega - \lambda) - d_0\,|^2 \le 0\,, \qquad \forall\,\omega \ge 0\,. \end{aligned} \tag{10}$$

(A6) The function $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and $\phi(t,0) = 0,\ \forall\, t \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$. There are numbers $\kappa_1 > 0$ (from **(A5)**), $0 \le \kappa_2 < \kappa_3 < +\infty, \beta_1 < \beta_2$ and $\zeta_2 < \zeta_1$ such that:

$$a)\quad \beta_1 < g(t) < \beta_2\,, \tag{11}$$

for a.a. t from an arbitrary compact time interval ;

$$\begin{aligned} &b)\quad (\phi(t,z) + \beta_i)(z - \zeta_i) \le \kappa_1(z - \zeta_i)^2, \quad i = 1,2\,, \\ &\qquad \forall\, t \in \mathbb{R}, \quad \forall\, z \in [\zeta_2,\zeta_1]\,; \end{aligned} \tag{11a}$$

$$\begin{aligned} &c)\quad \kappa_2(z_1 - z_2)^2 \le (\phi(t,z_1) - \phi(t,z_2))(z_1 - z_2) \le \\ &\kappa_3(z_1 - z_2)^2\,, \qquad \forall\, t \in \mathbb{R},\ \forall\, z_1,z_2 \in [\zeta_2,\zeta_1]\,. \end{aligned} \tag{11b}$$

We assume in the next theorem that the solutions of (1) are for every $T > 0$ elements of the space $\mathcal{W}(0,T;Y_1,Y_{-1})$. Then we show the existence of solutions with initial states from a certain set.

Theorem 1. Assume that for system (1) the hypotheses **(A1)** – **(A6)** are satisfied. Then there exists a closed, positively invariant and convex set $\mathcal{G}$ such that

$$\begin{aligned} &\{(\nu,z) \in \mathcal{V}_1\times\mathbb{R}\,|\,\nu = 0, z \in [\zeta_2,\zeta_1]\} \subset \mathcal{G} \subset \\ &\{(\nu,z) \in \mathcal{V}_1\times\mathbb{R}\,|\,z \in [\zeta_2,\zeta_1]\}\,. \end{aligned} \tag{12}$$

In order to prove this Theorem we need some auxiliary results. The full proof of Theorems 1 – 3, which will be published elsewhere, is based on the frequency theorem Likhtarnikov and Yakubovich [1976], Yakubovich [1964]. A similar approach was used in Reitmann [2005], Reitmann and Kantz [2004].

Suppose that $Y_1 \subset Y_0 \subset Y_{-1}$ is a Gelfand rigging of $Y_0, \|\cdot\|_j, (\cdot,\cdot)_j$ are the corresponding norms and scalar products, respectively, and $(\cdot,\cdot)_{-1,1}$ is the pairing between Y_{-1} and Y_1. Consider the linear system

$$\dot y = Ay\,, \quad z = (c,y)_0\,, \tag{13}$$

where $A \in \mathcal{L}(Y_1,Y_{-1})$ and $c \in Y_0$.

Assume that for each $y_0 \in Y_0$ there exists a unique solution $y(\cdot,y_0)$ of (13) in $\mathcal{W}(0,\infty)$ satisfying $y(0,y_0) = y_0$. In the sequel we need the following assumption Brusin [1976].

(A7) The space Y_0 can be decomposed as $Y_0 = Y_0^+ \oplus Y_0^-$ such that the following holds:

a) For each $y_0 \in Y_0^+$ we have $\lim_{t\to\infty} y(t,y_0) = 0$. For each $y_0 \in Y_0^-$ there exists a unique solution $y_-(t) = y(t,y_0)$ of (13), defined on $(-\infty,0)$, such that $\lim_{t\to-\infty} y_-(t) = 0$ and $(c,y(t,y_0))_0 = 0,\ \forall\, t \ge 0$, if and only if $y_0 = 0$.

b) For each $y_0 \in Y_0^+$ the equality $(c,y(t,y_0))_0 = 0$, $\forall\, t \le 0$, holds if and only if $y_0 = 0$. For each $y_0 \in Y_0^-$