FREQUENCY-DOMAIN CRITERION FOR STABILITY OF OSCILLATIONS IN A CLASS OF NONLINEAR FEEDBACK SYSTEMS

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Abstract: The paper, describes the problem of stability of oscillations in nonlinear feedback systems. The concept of stability is defined in a way that makes the problem tractable using the absolute stability approach. The result is formulated in frequency domain and has the form of the Zames-Falb multiplier, which makes it amenable to geometric interpretation. Numerical examples are given to illustrate the application of the new result to cases, where the Circle Criterion is not applicable. The advantage of the new criterion is that only the period of the oscillations needs to be known, not the complete expression of the oscillatory solution.

Keywords: Absolute stability, stability analysis, nonlinear control systems, oscillators.

1. INTRODUCTION

The problem of stability of periodic motion was first formulated in the classic book by Lyapunov (1992) and has since received considerable attention from many researchers, including Lyapunov himself.

The standard approach to this problem involves the investigation of the so-called variation equation. For the local stability problem, this approach leads to the well-studied linear differential equations with periodic coefficients (Yakubovich and Starzhinskii, 1975). The stability criteria involve computation of the Floquet multipliers, and various ad hoc estimation techniques. They are applicable to both forced and autonomous oscillations.

Another approach involves the use of the fixed-point theory. It is described in some length in the monographs by Holtzman (1974) and Burton (2005). Topological methods are studied in the book by Krasnoselskii (1968). These methods are applicable only to forced oscillations.

The approach proposed in this paper differs from the previous work in several ways. First, the nonlinear variation equations are studied instead of the linearized ones, leading to global, as opposed to local, stability results. Secondly, the proposed approach uses the known absolute stability criteria and, therefore, does not require any information about the periodic solution except for its period. Finally, the resulting criteria are much easier to check than the standard ones involving Floquet multipliers found in many standard textbooks on differential equations. The results are applicable to both forced and autonomous oscillations.

2. FORMULATION OF THE PROBLEM

Our starting point is the nonlinear feedback system in the vector-matrix form:

$$\dot{x} = Ax + Bu + f(t) \tag{1}$$

$$y = Cx \tag{2}$$

$$u = \varphi(y)\,. \tag{3}$$

Here *f(t)* is a periodic vector function with the period *T*. It may be identically equal to zero, but then there is a question of the existence of periodic solutions, which is outside of the scope of this paper. The function For the sake of simplicity, we consider the case of a SISO system, i.e. both the function $\varphi(\sigma)$, hereafter called the nonlinearity, and its argument σ are scalar. The results can be easily extended to MIMO systems.

We now proceed to the formulation of the results.

3. STATEMENT OF THE RESULTS

The main results of this paper follow almost directly from the earlier absolute stability results for systems with time periodic nonlinearities. For this reason they will be stated without proof. Throughout the paper, we denote the transfer function of linear part of the system by *W(s)* and define it by the usual equation:

$$W(s) = C^*[sI - A]^{-1}B\ .$$

3.1 Analytic Criterion.

The following result is an immediate consequence of the Theorem 3.2.5 from (Altshuller, 2004) and restated for sake of completeness in the Appendix.

For the sake of brevity we introduce the following notation: For any function $X(i\omega)$, the expression $\operatorname{Re} X(i\omega) >> 0$ means that there exists a constant $\delta > 0$, such that for any real number ω, $\operatorname{Re} X(i\omega) > \delta$.

THEOREM 1. Suppose that:

1. *The matrix* A *is Hurwitz;*
2. *For all* t *and all* $\sigma_1 \neq \sigma_2$;

$$0 \le \frac{\varphi(\sigma_1,t) - \varphi(\sigma_2,t)}{\sigma_1 - \sigma_2} \le \mu < \infty\ ;$$

3. *There exists a sequence* θ_n *with nonnegative terms, such that* $\sum_{n=1}^{\infty}\theta_n < 1$ *and*

$$\operatorname{Re}\left\{\left[\mu^{-1} + W(i\omega)\right]Z(i\omega)\right\} >> 0 \tag{4}$$

with

$$Z(i\omega) = 1 - \sum_{n=0}^{\infty}\theta_n e^{i\omega nT}\ . \tag{5}$$

Let $y = \phi(t)$ *be the output of the system (1-3) having a period* T. *Then for the output* $y = \bar{\phi}(t)$ *corresponding to any other solution of the system (1-3)*

$$\sigma(\cdot) \in L_2(0,\infty)$$

and there exists a constant λ *, independent of the function* $\alpha(\cdot)$*, such that*

$$\|\sigma(\cdot)\| \le \lambda \|\alpha(\cdot)\|,$$

where $\sigma(t) = \bar{\phi}(t) - \phi(t)$.

The general nature of this criterion makes difficult to use since it is not clear how to find the desired sequence θ_n. However, the left-hand side of the inequality (4) has a very convenient Zames-Falb multiplier form, which makes it possible to interpret this criterion geometrically as we proceed to do in the next subsection.

3.2 Geometric Interpretation.

With a slight abuse of notation, we can rewrite the inequality (4) in the form:

$$[\mu^{-1} + \operatorname{Re} W(i\omega)]\operatorname{Re} Z(i\omega) - \operatorname{Im} W(i\omega)\operatorname{Im} Z(i\omega) > 0$$

Let us define the two functions:

$$\Phi(\omega) = \frac{\mu^{-1} + \operatorname{Re} W(i\omega)}{\operatorname{Im} W(i\omega)},\ \ \Psi(\omega) = \frac{\operatorname{Im} Z(i\omega)}{\operatorname{Re} Z(i\omega)}\,.$$

Note that since $\operatorname{Re} Z(i\omega) > 0$, the function $\Psi(\omega)$ is continuous for all values of ω.

It is relatively easy to show (Lipatov, 1981) that for the type of systems under consideration the graph of the function $\Phi(\omega)$ consists of branches with asymptotes. The ends of the branches point either to $+\infty$ (Such branches are called stalactites) or to $-\infty$ (Such branches are called stalagmites). The inequality (4) holds if a function $\Psi(\omega)$ can be found such that its graph separates stalactites from the stalagmites.

This geometric interpretation has been used extensively for systems with stationary nonlinearities. For the time-dependent case, the best known result is the Circle Criterion, for which $\Psi(\omega) \equiv 0$.

For the expression given by the Equation (5) we have:

$$\Psi(\omega) = \frac{\sum_{n=0}^{\infty}\theta_n \sin\omega nT}{\sum_{n=0}^{\infty}\theta_n \cos\omega nT - 1}\,.$$

This geometric interpretation of the analytic criterion is easier to use if the infinite series are replaced with finite sums. In the next section, several numerical examples will be given to illustrate the application of the criterion.