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A simple formula for quad mesh singularities

Harold J. Fogg[a], Liang Sun[b], Jonathan E. Makem[a], Cecil G. Armstrong[b], Trevor T. Robinson[b]

[a]*Meshing & Abstraction, Digital Factory, Simulation and Test Solutions Siemens PLM Software, SIEMENS, Francis House, 112 Hills Road, Cambridge, UK. CB2 1PH*

[b]*School of Mechanical and Aerospace Engineering, Queen's University Belfast, UK. BT9 5AH*

Abstract

A formula is presented for determining the net sum of mesh singularity indices that must occur in an all-quadrilateral (quad) mesh of a face or surface region after the mesh properties have been assigned on the face's boundaries and according to the face's Euler Characteristic. The formula is derived from Bunin's *Continuum Theory for Unstructured Mesh Generation* [1].

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1. Introduction

Structured grids with tensor product structure offer numerical advantages over unstructured meshes. However, when it comes to closed surfaces we know from such familiar things as globes that sometimes singularities, where the regular grid structure is disrupted, are unavoidable. When the surface has boundaries forcing the grid to conform to them can also necessitate the introduction of singularities, not just to reduce the distortion of the grid, but there will be a certain minimum number that are essential for facilitating an all-quad mesh. How to determine the requisite mesh singularities based on the topological shape of the surface region and the geometric shapes of its boundaries will be covered in this paper.

1.1. Related work

A formula was proposed by White *et al.* [2] for determining if a face is submappable, which means that a *logical* representation can be found where all edges of the face are horizontal or vertical. First vertices are assigned types, which essentially decides the number of $\pi/2$ turns (signed with respect to an anticlockwise traversal) between the adjacent edges in a local logical representation. Values of 1, 0, -1 and -2 are assigned to *end*, *side*, *corner* and *reversal* vertices (*cf.* Fig. 1). If the sum of the vertex classification values is four it means that the local logical representations

[a] harry.fogg@siemens.com
[b] Liang.Sun@qub.ac.uk

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of the vertices are consistent with a global logical representation for the face and thus the face is submappable. The check works on the principle that the sum of exterior angles at vertices of a planar polygon must equal 2π.

The formula was reformulated and generalised to multiply connected faces by Ruiz Girones *et al.* [3]. For the face to be submappable it must have assigned vertex types such that

$$E - C - 4R = 4(1 - H) \tag{1}$$

where E, C and R are the number of *end*, *corner* and *reversal* vertex types and H is the number of inner boundary loops.

Parametrisation-based mesh generation techniques have been intensely developed recently [4,5]. Their approach is to initialise and optimise a rotational symmetry vector field to improve its smoothness and in the process establish the number, indices and placements of mesh singularities. A secondary optimisation method solves for a parametrisation and hence mesh that fits with the rotational symmetry vector field. Ray *et al.* [6] and Knöppel *et al.* [7] prove the Poincaré-Hopf theorem on smooth and discrete closed oriented surfaces with empty boundaries which relates the net sum of singularity indices (defined slightly differently to here) to its Euler Characteristic.

1.2. Contributions

A new formula is presented which is equally simple but is more descriptive to that of Ruiz Girones *et al.*. It determines the net sum of mesh singularity indices that are required for a face, not just whether the face is submappable, and it accounts for general genera. The result is analogous to the Poincaré-Hopf theorem with the extension to non-empty boundaries on which alignment constraints are enforced. The particular case of four-way rotational symmetric direction (cross-) fields is specifically dealt with although the result could be extended to N-way rotational symmetric direction fields. The sufficiency of the formula for all-quad meshes is outlined.

2. Predicating theories

2.1. Euler Characteristic

If the face R is a regular (*i.e.* compact orientable) region of surface S its Euler Characteristic can be defined by

$$\chi(R) = V - E + F, \tag{2}$$

where V, E, F are the number of vertices, edges and facets of a triangulation (or a polygon mesh) on the face. The value does not depend on triangulation, hence it *characterises* the face [8, Prop. 1, Chp. 4.5]. The Euler Characteristic of a simple face R = disc can be ascertained by analysing a single triangle:

$$V = 3, E = 3, F = 1 \rightarrow \chi(\text{disc}) = 1. \tag{3}$$

The relationship between the genus, $g(S)$, and Euler Characteristic, $\chi(S)$, of a closed orientable surface with empty boundary S is

$$g(S) = \frac{2 - \chi(S)}{2} \tag{4}$$

[8, Prop. 4, Chp. 4.5]. The table to the right gives the genera and Euler Characteristics of familiar closed surfaces.

S	$g(S)$	$\chi(S)$
sphere	0	2
torus	1	0
double torus	2	-2
⋮	⋮	⋮

2.2. Global Gauss-Bonnet Theorem

The Global Gauss-Bonnet Theorem [8, Chp. 4.5] is

$$\sum_{i=1}^{N} \int_{C_i} \kappa_g(S)\, ds + \iint_R K(S)\, dA + \sum_{j=1}^{N} \gamma_j = 2\pi\chi(R) \tag{5}$$