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Nonlinear Fourier Methods for Ocean Waves

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Abstract

Multiperiodic Fourier series solutions of *integrable nonlinear wave equations* are applied to the study of ocean waves for scientific and engineering purposes. These series can be used to compute analytical formulae for the *stochastic properties* of nonlinear equations, in analogy to the standard approach for linear equations. Here I emphasize analytically computable results for the *correlation functions*, *power spectra* and *coherence functions* of a *nonlinear random process* associated with an integrable nonlinear wave equation. The multiperiodic Fourier series have the advantage that the *coherent structures* of soliton physics are encoded in the formulation, so that *solitons*, *breathers*, *vortices*, etc. are contained in the *temporal evolution* of the nonlinear power spectrum and phases. I illustrate the method for the Korteweg-deVries and nonlinear Schrödinger equations. Applications of the method to the analysis of data are discussed.

Keywords: nonlinear integrable wave equations; finite gap theory; periodic inverse scattering transform; nonlinear Fourier analysis; nonlinear ocean waves; nonlinear numerical methods; nonlinear time series analysis.

1. Multiperiodic Fourier Series as Solutions of Nonlinear Integrable Wave Equations

I give an overview of *multiperiodic Fourier series solutions of integrable, nonlinear wave equations*. I discuss how these series can be used as practical tools for the study of nonlinear ocean waves in one and two dimensions. This perspective has arisen from the pure mathematical algebraic-geometric construction of *single valued, multiperiodic, meromorphic Fourier series* from *Riemann theta functions* [1-3, 23-24, 29, 32]. Modern interest in this area of research has occurred because recent developments have shown how theta functions can be used to solve *nonlinear integrable wave equations* [4, 24], a field known as *finite gap theory* (FGT) or the *periodic inverse scattering transform*. Many of these developments have been used for oceanographic

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applications [8, 28]. Here I address how multiperiodic, meromorphic Fourier series constructed from theta functions can themselves also be generically useful as mathematical and data analysis tools. The resultant formulation I refer to as *nonlinear Fourier analysis*, where the *Stokes wave* is a single degree of freedom component of the nonlinear Fourier theory, as opposed to the sine wave of linear Fourier series. *Coherent structures* such as solitons, breathers and vortices are all constructed naturally from the Stokes waves by increasing the nonlinearity and/or by phase locking two Stokes components. Nonlinear interactions amongst the Stokes waves are also formulated. The nonlinear Fourier methods are applicable to many aspects of the study of ocean waves from scientific, engineering, numerical, and data analysis perspectives. Sections 1 and 2 give some historical perspective, while Sections 3-6 discuss applications of physically relevant nonlinear wave equations.

Why do I take the path of periodicity/quasiperiodicity offered by FGT for solving the *spectral structure* of *nonlinear wave equations* describing ocean waves? Why would this approach lead to *multiperiodic Fourier series* for their description rather than ordinary *trigonometric Fourier series*? Here are a few of the reasons:

(1) To analyze *time series data* one most often assumes the data to be *periodic* and *discrete*. The fast Fourier transform (FFT), mathematically a discrete Fourier transform, is used for numerical computations.
(2) To develop a natural theory for the *nonlinear Fourier analysis of wave motion* founded on *Stokes wave basis functions* that *interact nonlinearly* with each other.
(3) To have *data analysis* and *analytical approaches* that include *coherent structures in the nonlinear Fourier analysis*. These include *Stokes waves, solitons, breathers and vortices*.
(4) To develop a full theory of *nonlinear random wave trains* for describing *stochastic ocean waves for nonlinear integrable systems*.
(5) To develop a method which naturally extends to *nonintegrability of perturbed (or higher order) nonlinear systems by finding ordinary differential equations that vary adiabatically in the Riemann spectrum of the solitons, breathers, etc*. This is because nearly integrable systems can often be treated with the slow time evolution of their FGT spectra.

An introduction to some of this material is already presented in Osborne (2010), but much of the work presented here is a new approach for *nonlinear*, *stochastic ocean waves*, in which the *correlation function* and *power spectrum*, together with other stochastic properties, can be computed analytically from the multiperiodic series.

Anticipating later results below, we consider the possibility that there exist *spectral solutions* of *integrable, nonlinear wave equations* that have the form of *multidimensional, quasiperiodic Fourier series*

$$u(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{1}$$

We will see that these are regarded, from the point of view of algebraic geometry, as the *most general, single valued, multiply periodic meromorphic functions of N variables with 2N periods (Baker-Mumford Theorem)*. Here the wavenumbers $\mathbf{k}$, the frequencies $\boldsymbol{\omega}$ and the phases $\boldsymbol{\phi}$ are vectors of dimension N (the genus). The summation index $\mathbf{n}$ is over the integer lattice $\mathbb{Z}^N$. Eq. (1) can be written as a nested summation:

$$u(x,t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} u_{n_1,n_2\dots n_N} e^{i\sum_{i=1}^{N} n_i k_i x - i\sum_{i=1}^{N} n_i \omega_i t + i\sum_{i=1}^{N} n_i \phi_i} \tag{2}$$

I discuss below how series of this type can be constructed with the aid of the *Baker-Mumford Theorem* and from *finite gap theory using theta functions*. But first I consider a number of properties of these series.

1.1. Reduction of Multiperiodic Fourier Series to Ordinary Trigonometric Series

Assume the solution of an integrable, nonlinear partial differential equation is given by (1). For *spatially periodic boundary conditions* $u(x,t) = u(x+L,t)$ the multidimensional, quasiperiodic Fourier series (1) can be written as a trigonometric series with time varying coefficients [28]:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{ik_n x}, \qquad k_n = \frac{2\pi n}{L} \tag{3}$$

$$u_n(t) = \sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ k_n = \mathbf{n}\bullet\mathbf{k}\}} u_{\mathbf{n}} e^{-i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}}, \qquad \mathbf{n}\bullet\mathbf{k} = \sum_{i=1}^{N} n_i k_i \tag{4}$$