EXPONENTIAL ESTIMATES FOR TIME DELAY SYSTEMS USING FUNCTIONALS WITH DERIVATIVE OR DELAYED TERMS

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Abstract: In this contribution, we explain that using Lyapunov Krasovskii functionals with time derivative terms for the determination of exponential estimates of the system response restrict the scope of the results. We show that it is possible to overcome this limitation by introducing delayed states in the functional. The case of linear time delay systems of retarded type is analyzed and it is shown that our results match or improve existing results in the literature.

Keywords: time delay systems, Lyapunov Krasovskii functional, exponential estimates

1. INTRODUCTION

In the last couple of decades, a large amount of literature on the stability of delay dependent linear time delay systems have been reported in the framework of the Lyapunov Krasovskii approach. Although this theory (Krasovskii, 1956) establishes the existence of a functional for any stable linear time delay system (see among others, (Infante and Castelan, 1978) and (Kharitonov and Zhabko, 2003)), its design is computationally demanding. This is why the design of simple functionals that lead to sufficient conditions expressed in terms of easily checkable Linear Matrix Inequalities remains very popular. (for a complete review, see (Niculescu, 2001), (Gu and Niculescu, 2003)). The main approaches for obtaining delay dependent conditions are the explicit model transformations method (Kolmanovskii and Myshkis, 1999) whose conservativeness, due to added dynamics, has been extensively studied in (Kharitonov and Melchor-Aguilar, 2002), (Gu and Niculescu, 2000), the descriptor model transformation method (Fridman and Shaked, 2002).

Recently, the introduction of derivative terms in the Lyapunov-Krasovskii functionals (Park, 1999), (Mahmoud, 1998), (He *et al.*, 2004), (Moon *et al.*, 2001), (Xu and Lam, 2005) called sometime implicit model transformations, was shown to improve substantially previous conditions (Zhang *et al.*, 2001).

However, when the problem of determining exponential estimates of the system response is addressed, including derivative terms of the solution into the functional reduces the scope of the result to the particular class of solutions with derivable initial conditions. In this note, we show that it is possible to overcome this limitation by using instead functionals with delayed terms.

This contribution is organized as follows: in section 2, some results of the Lyapunov-Krasovskii approach are recalled as a background for the

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discussion of section 3. In section 4, these considerations lead to an exponential stability result for functionals with delayed terms. Next, in section 4, the particular case of linear systems is addressed. In section 5, we compare our results with previously published ones.

2. SOME BACKGROUND

For the sake of clarity, we recall next some definitions and classical results concerning the existence and stability of the solutions of general time delay systems of retarded type described by

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $x(t) \in R^n$ is the solution of the functional differential equation (1), and x_t denotes the segment of trajectory $x(t+\theta)$, $\theta \in [-h, 0]$ with $h \geq 0$ is the maximum delay. To determine the future evolution of the state, the variable $x(t)$ must be specified in a time interval of length h, say from $t_0 - h$ to t_0, i.e., $x_{t_0} = \varphi$,in other words, $x(t_0 + \theta) = \varphi(\theta)$, $\theta \in [-h, 0]$. When the dependence on initial condition φ is relevant, the solution and the state variable at time t are denoted $x(t, \varphi)$ and $x_t(\varphi)$, respectively. Unless otherwise specified, the space of initial functions is provided with the uniform continuous norm

$$\|\varphi\|_h = \max_{\theta \in [-h,0]} \{\|\varphi(\theta)\|\}, \tag{2}$$

where $\|\cdot\|$ stands for the Euclidean norm for vectors.

Assumption A:

The functional f is defined for $t \geq 0$ and $\|x_t\|_h \leq H$. It is continuous and Lipschitzian with respect to the second argument, i.e., there exist a constant $L > 0$ such that

$$\|f(t, \varphi_1) - f(t, \varphi_2)\| \leq L \|\varphi_1 - \varphi_2\|_h .$$

In the sequel, we assume also with no loss of generality that (1) admits the solution $x_t \equiv 0$ which is called the trivial solution, and that the initial time is $t_0 = 0$.

First, we recall the sufficient conditions for the existence of solutions for this system.

Theorem 1. (Bellman and Cooke, 1963) Consider the system (1) where f satisfies Assumption A. Then, for any continuous initial function $\varphi \in C([-h, 0], R^n)$, there exists a unique solution $x(t, \varphi)$ of the system (1) satisfying the initial condition

$$x(\theta) = \varphi(\theta), \quad \theta \in [-h, 0].$$

An important result is also that if f in system (1) satisfies Assumption A, it is also possible to establish with the help of the Bellman Gronwall Lemma that these solutions are of exponential order.

Lemma 2. (Bellman and Cooke, 1963) Consider the system (1) where f is such that Assumption A holds. Then, for any continuous initial function $\varphi \in C([-h, 0], R^n)$, the unique solution of system (1) satisfies

$$\|x_t(\varphi)\|_h \leq e^{Lt} \|\varphi\|_h, \quad t \geq 0. \tag{3}$$

This fundamental result that holds in particular for linear systems is important, because it means that when one addresses the problem of stability, one can with a marginal additional effort obtain exponential estimates for the decay rate σ and an upper bound for the norm γ of the solution of system (1) that are formally defined as follows.

Definition 1. The system (1) is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma \geq 1$ such that for every solution $x(t, \varphi)$, where $\varphi \in C([-h, 0], R^n)$, the following exponential estimate holds

$$\|x(t, \varphi)\| \leq \gamma e^{-\sigma t} \|\varphi\|_h, \quad \forall t \geq 0.$$

These estimates can be obtained with the help of the following fundamental result

Theorem 3. Suppose that f in system (1) satisfies Assumption A and let α_1, α_2, and β be positive constants. If there exists a continuous differentiable functional $V(t, x_t)$ that maps $[h, \infty)\times$ bounded sets of $C([-h, 0], R^n)$ into bounded sets of R such that

$$\alpha_1 \|x(t)\|^2 \leq V(t, x_t) \leq \alpha_2 \|x_t\|_h^2, \quad \forall t \geq 0 \tag{4}$$

and

$$\dot{V}(t, x_t) + 2\beta V(t, x_t) \leq 0, \quad \forall t \geq 0, \tag{5}$$

then, for any initial condition φ in $C([-h, 0], R^n)$ exponential estimates for the unique solution of (1) are given by the inequality

$$\|x(t, \varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\varphi\|_h e^{-\beta t}, \quad t \geq 0. \tag{6}$$

Proof. Observe that (5) implies that

$$\frac{d}{dt}\left(e^{2\beta t} v(t, x_t)\right) \leq 0, \quad t \geq 0.$$

This inequality leads to the following one

$$V(t, x_t(\varphi)) \leq e^{-2\beta t} V(0, x_0(\varphi)), \quad t \geq 0$$

and the inequalities (4) provide the estimations

$$\begin{aligned}\alpha_1 \|x(t, \varphi)\|^2 &\leq V(t, x_t(\varphi)) \\ &\leq e^{-2\beta(t-h)} V(h, x_h(\varphi)) \\ &\leq \alpha_2 e^{-2\beta t} \|x_h(\varphi)\|_h^2, \quad t \geq 0.\end{aligned}$$