MAPPING THE SPECTRUM OF A RETARDED TIME-DELAY SYSTEM UTILIZING ROOT DISTRIBUTION FEATURES

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Abstract: An original method for computing the spectrum of a retarded time delay system over a large region in the complex plane is presented. The method is based on mapping the quasi-polynomial characteristic function and on utilizing asymptotic properties of the root chains. First, asymptotic exponentials of the root chains are assessed and large areas free of the roots are determined. These areas are then omitted in systematic mapping of the roots location. The roots are located as the intersection points of the contours determined by real and imaginary parts of the characteristic function. The theoretical explanation of the method is supplemented with a discussion of implementation issues and an application example is added.

Keywords: retarded system, quasi-polynomial, spectrum distribution, root-finding techniques

1. INTRODUCTION

Investigating the dynamic features of time delay systems (TDS) in the frequency domain is, mainly due to infinite dimensionality of TDS, a challenging issue. Even though many analytical methods for such investigation have been developed (Kolmanovskii and Myshkis, 1992), (Hale and Verduyn Lunel, 1993), due to complexity of the problem, their applicability is often restricted to a narrow class of systems (e.g. TDS with a single delay). To overcome the restrictions of the analytical approaches, many of recently developed algorithms combine numerical and analytical tools. The algorithm for computing all the roots of TDS located in large regions of the complex plane described in this paper follows this novel stream of the dynamics analysis of TDS. First, Bellman and Cooke (1963) analytical approach for determining the strips in the complex plane where the root chains are located is utilized to reduce significantly the area of the domain to be scanned for the roots. Consequently, the location of the roots in the strips is determined using mapping based method designed earlier by the authors (Vyhlídal and Zítek, 2003).

Due to space limitation, the stress is laid on the implementation issues. Missing proofs and discussion on the theoretical issues can be found in the extended version of this article (Vyhlídal and Zítek, 2006).

2. PROBLEM STATEMENT

Consider TDS of retarded type

$$\mathbf{x}'(t) = \sum_{j=0}^{\nu} \mathbf{A}_j \mathbf{x}(t-\tau_j) \qquad (1)$$

where $\mathbf{x} \in \mathrm{R}^n$ is the vector of state variables, $\mathbf{A}_j \in \mathrm{R}^{n\times n}$, $j=0..\nu$, are the constant coefficient matrices and $0=\tau_0<\tau_1<\ldots<\tau_{\nu-1}<\tau_\nu$ are the time delays. The characteristic equation of system (1) is given by

$$h(s) = \det(s\mathbf{I} - \sum_{j=0}^{\nu} \mathbf{A}_j \mathrm{e}^{-s\tau_j}) = \sum_{j=0}^{N} p_j(s)\mathrm{e}^{-s\alpha_j} = 0 \qquad (2)$$

where $\alpha_0>\alpha_1>\ldots>\alpha_{N-1}>\alpha_N=0$, each α_j is a combination of $\tau_0,\tau_1,\ldots,\tau_\nu$ and each $p_j(s)=p_{j0}+p_{j1}s+\ldots+p_{jm_j}s^{m_j}$ is a polynomial in s of degree at most n.

Due to a quasi-polynomial form of function $h(s)$, the characteristic equation (2) is transcendental and it has infinitely many roots. Since there are no general analytical formulas available for computing the zeros of any quasi-polynomial of form (2), a numerical method has to be used.

3. ON COMPUTING SPECTRUM OF TDS

The methods which are usually used for computing the spectrum of TDS are based on approximation either of the system solution operator or its infinitesimal generator. In both approaches, the rightmost part of the spectrum of TDS is approximated by the eigenvalues of a large dimensinal matrix resulting from the approximation. In (Banks and Kappel, 1979), (Kazufomi 1985), the problem is solved via approximating the infinitesimal generator using spline functions. In (Breda et al., 2004), see also the references therein, the generator is approximated using Runge-Kutta methods. In (Engelborghs and Roose, 2002) Linear Multi-Step methods are used for the approximation (the method is implemented in DDE-Biftool Matlab package, see (Engelborghs, et al., 2002)). In (Breda, 2005), the solution operator is approximated using Runge-Kutta methods. It is well known that the mentioned methods approximate well only the spectrum of rightmost roots, i.e. the roots with smallest modulus. On the other hand, the methods are fast and numerically reliable.

Alternatively, the spectrum of system (1) can be computed by a numerical algorithm developed for computing zeros of general analytic functions, e.g. the methods based on bisection, (Dellnitz, et al. 2002) or quadrature methods (Delves and Lyness, 1967), (Kravanja and Van Barel, 2000), or methods based on iteration schemes, e.g. Newton's method.

In their earlier work, the authors introduced an algorithm to compute the spectrum of a TDS located in a region $\mathfrak{D}$ of the complex plane (Vyhlídal and Zítek, 2003). The algorithm, which is based on mapping the quasi-polynomial, is easy to implement and proved efficient in locating large number of roots. The algorithm will be explained in the next section.

4. MAPPING BASED ALGORITHM

The basic idea of the quasi-polynomial mapping based rootfinder (QPMR) is given as follows. Consider functions $R(\beta,\omega) = \mathrm{Re}(h(\beta+\mathrm{j}\omega))$ and $I(\beta,\omega) = \mathrm{Im}(h(\beta+\mathrm{j}\omega))$. Then the characteristic equation (2) can be split into

$$R(\beta,\omega) = 0 \tag{3}$$

$$I(\beta,\omega) = 0 \tag{4}$$

These equations determine the intersection contours of the surfaces described by $R(\beta,\omega)$ and $I(\beta,\omega)$, respectively, with the s-plane. Mapping these zero-level contours, the roots of (2) are given as the intersection points of contours described implicitly by (3) and (4). Obviously, the contours $R(\beta,\omega) = 0$ and $I(\beta,\omega) = 0$ can be expressed analytically only for the most simple quasi-polynomials, see (Vyhlídal and Zítek, 2006). In general, a numerical contour plotting algorithm is to be used to map the contours, e.g., the level curve tracing algorithm (implemented, e.g. in Matlab function *contour*). In the practical implementation, first, the region of interest in the complex plane $\mathfrak{D}$ is covered by a regular mesh grid (with a grid step Δ_g) and the function $R(\beta,\omega)$ ($I(\beta,\omega)$) is evaluated at each point of such a grid. Then, the zero-level contours are constructed using contour plotting algorithm. The mapping procedure and the algorithm for determining the intersection points of the contours are described in detail in (Vyhlídal and Zítek, 2003).

A considerable drawback of the QPMR is the need to evaluate the functions $R(\beta,\omega)$ and $I(\beta,\omega)$ at each point of the grid spread over the region $\mathfrak{D}$. Thus, if the scanned region $\mathfrak{D}$ is large and Δ_g small, the computation can become unacceptably time consuming. In order to reduce considerably duration of the computation, the features of the spectrum of a retarded system will be utilized to reduce the area which is to be scanned for the roots. Since the spectrum of a retarded system consists of a finite number of asymptotic chains, there exist large areas in the complex plane free of roots. Further, we propose a procedure for identifying these areas which are then omitted from searching for the roots.

5. ASYMPTOTIC DISTRIBUTION OF THE ROOT CHAINS

In this section we adopt the results of Bellman and Cooke (1963) concerning the distribution of the roots with large modulus. Let the characteristic equation (2) be multiplied by $\mathrm{e}^{s\alpha_0}$, then

$$g(s) = h(s)\mathrm{e}^{s\alpha_0} = \sum_{j=0}^{N} p_j s^{m_j}(1+\varepsilon_j(s))\mathrm{e}^{s\vartheta_j} = 0 \tag{5}$$

where $\vartheta_j = \alpha_0 - \alpha_j$, $0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_{N-1} < \vartheta_N$, $p_j \neq 0\ (j = 0,1,\ldots,N)$ and the function $\varepsilon_j(s)$ have the property $\lim_{|s|\to\infty} |\varepsilon_j(s)| = 0$. Since the term $\mathrm{e}^{s\alpha_0}$ has no zeros, equations (2) and (5) have the same distribution of roots.

As it has been designed in (Bellman and Cooke, 1963), with the points $P_j = (\vartheta_j, m_j)$, we can define the *spectrum distribution diagram* (also called potential diagram, (Gu, et al., 2003)) of (5).