A LYAPUNOV ISS SMALL-GAIN THEOREM FOR STRONGLY CONNECTED NETWORKS

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Abstract: We consider strongly connected networks of input-to-state stable (ISS) systems. Provided a small gain condition holds it is shown how to construct an ISS Lyapunov function using ISS Lyapunov functions of the subsystems. The construction relies on two steps: The construction of a strictly increasing path in a region defined on the positive orthant in $\mathbb{R}^n$ by the gain matrix and the combination of the given ISS Lyapunov functions of the subsystems to a ISS Lyapunov function for the composite system.
Novelties are the explicit path construction and that all the involved Lyapunov functions are nonsmooth, i.e., they are only required to be locally Lipschitz continuous. The existence of a nonsmooth ISS Lyapunov function is qualitatively equivalent to ISS.

Keywords: Lyapunov function, nonlinear control systems, nonlinear gain, stability criteria, decentralized systems.

1. INTRODUCTION

In this paper we are interested in the stability of a network of nonlinear input to state stable (ISS) systems. A nonlinear small gain theorem for networks of input-to-state stable (ISS) systems was obtained in Dashkovskiy et al. (2007). Here we provide a constructive method to find a nonsmooth ISS Lyapunov function for a composite system, when the ISS Lyapunov functions and nonlinear gains for the subsystems are all known. This result is particularly useful, since the knowledge of a Lyapunov function directly leads to knowledge of invariant sets and allows for different controller design methods, see, e.g., Khalil (1996). A main step of the construction was already carried out in Dashkovskiy et al. (2006c). Namely, it was shown how to construct a nonsmooth ISS Lyapunov function, if a strictly increasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ exists such that $D(\Gamma(\sigma(t))) < \sigma(t)$ for all $t > 0$. Here Γ is the gain matrix, and D is a diagonal scaling operator. In Dashkovskiy et al. (2006c) the existence of such a function was shown only for the case of three interconnected systems. The case of two systems in a feedback loop was considered in Jiang et al. (1994) and the construction of Lyapunov functions for this case was presented in Jiang et al. (1996).

The small gain condition derived in Dashkovskiy et al. (2007) leads to interesting invariance properties of the map defined by Γ, which allow a construction of the desired σ. Here we are going to construct a σ, that is differentiable almost everywhere. The overall Lyapunov function is then obtained as a weighted maximum of the ISS Lyapunov functions of the subsystems similar to Jiang et al. (1996). As a consequence the constructed Lyapunov function is not differentiable, so that we resort to nonsmooth formulations of ISS Lyapunov functions. An alternative would be to use a smooth approximation, which is possible in principle. We avoid this as it does not add to the understanding of our construction.

In Proposition 12 we construct a piecewise linear and strictly increasing function $\sigma_s : [0,1] \to \mathbb{R}^n_+$ up to some predetermined radius, provided that Γ is irreducible. If Γ is even primitive, then this function can be extended to a function $\sigma \in \mathcal{K}^n_\infty$. If Γ is only irreducible, this function σ can still be defined, but under slightly stronger assumptions, see Theorem 14.

2. NOTATION

Let $\mathcal{K} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ : f$ is continuous, strictly increasing and $f(0) = 0\}$ and $\mathcal{K}_\infty = \{f \in \mathcal{K} : f$ is unbounded$\}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first component and strictly decreasing to zero in the second component.

A matrix $\Gamma = (\gamma_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ defines a map on $\mathbb{R}^n_+$ via $\Gamma(s)_i = \sum_{j=1}^n \gamma_{ij}(s_j)$, for $s \in \mathbb{R}^n_+$, in analogy to matrix vector multiplication.

The adjacency matrix $A_\Gamma = (a_{ij})$ of a matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is defined by $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$ and $a_{ij} = 1$ otherwise. We say that the matrix Γ is *primitive, irreducible* or *reducible* if and only if A_Γ is primitive, irreducible or reducible. See e.g. Berman and Plemmons (1979) for definitions.

On $\mathbb{R}^n_+$ we use the partial order induced by the positive orthant. For vectors $x, y \in \mathbb{R}^n_+$ we define

$$x \geq y :\iff x_i \geq y_i \text{ for } i = 1,\dots,n,$$
$$x > y :\iff x_i > y_i \text{ for } i = 1,\dots,n, \text{ and}$$
$$x \gneqq y :\iff x \geq y \text{ and } x \neq y.$$

A map $\Delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *monotone* if $x \leq y$ implies $\Delta(x) \leq \Delta(y)$. Clearly $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ induces a monotone map. For $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $\Delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we write $\Gamma \geq \Delta$ if for all $x \in \mathbb{R}^n_+$ we have $\Gamma(x) \geq \Delta(x)$. Similarly, we write $\Gamma \not\geq \Delta, \Gamma > \Delta$, respectively $\Gamma \gneqq \Delta$, if for all $x \in \mathbb{R}^n_+ \setminus \{0\}$ we have $\Gamma(x) \not\geq \Delta(x)$, $\Gamma(x) > \Delta(x)$, respectively $\Gamma(x) \gneqq \Delta(x)$. Here $x \not\geq y$ means that for at least one component i the inequality $x_i < y_i$ holds.

For monotone maps Γ on $\mathbb{R}^n_+$ we define the following sets:

$$\Omega(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x) < x\},$$
$$\Omega_i(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x)_i < x_i\},$$
$$\Psi(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x) \leq x\}.$$

If no confusion arises we will omit the reference to Γ. Note that for general monotone maps we have $\overline{\Omega} \subsetneq \Psi$, but for $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ we have equality.

By $|\cdot|$ we denote the 1-norm on $\mathbb{R}^n$ and by S_r the induced sphere of radius r in $\mathbb{R}^n$ intersected with $\mathbb{R}^n_+$, which is an n-simplex. By $U_\varepsilon(x)$ we denote the open neighborhood of radius ε around x with respect to the Euclidean norm $\|\cdot\|$.

For our construction we will need the notions of proximal subgradient and nonsmooth ISS Lyapunov functions, c.f. Clarke et al. (1998), Clarke (2001). Also we need some results from nonsmooth analysis.

Definition 1. A vector $\zeta \in \mathbb{R}^N$ is a proximal subgradient of a function $\phi : \mathbb{R}^N \to (-\infty, \infty]$ at $x \in \mathbb{R}^N$ if there exists a neighborhood $U(x)$ of x and a number $\sigma \geq 0$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x\rangle - \sigma|y - x|^2 \quad \forall y \in U(x).$$

The set of all proximal sub-gradients at x is the proximal sub-differential of ϕ at x and is denoted by $\partial_P \phi(x)$.

3. INPUT-TO-STATE STABILITY

We consider a finite set of interconnected systems

$$\Sigma_i : \dot{x}_i = f(x_1, \dots, x_n, u), \; f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}, \tag{1}$$

$i = 1, \dots, n$, where $x_i \in \mathbb{R}^{N_i}$, $u \in \mathbb{R}^M$, $\sum N_i = N$.

If we consider one of the systems, indexed by i, and interpret the variables x_j, $j \neq i$, and u as unrestricted inputs, then this system is assumed to have unique solutions defined on $[0, \infty)$ for all L^∞-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_j}, j \neq i$, and $u : [0, \infty) \to \mathbb{R}^M$.

We write the interconnection of systems (1) as

$$\Sigma : \; \dot{x} = f(x, u), \quad f : \mathbb{R}^{N+M} \to \mathbb{R}^N, \tag{2}$$

where $x = (x_1^T, \dots, x_n^T)^T$.

We will impose ISS conditions on the subsystems given by (1) and we are interested in conditions guaranteeing ISS of the interconnected system (2). To this end we will construct an ISS Lyapunov function for (2).

Definition 2. (ISS Lyapunov function). A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an *ISS Lyapunov function* of (2) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{3}$$

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)). \tag{4}$$

The function χ is called a *Lyapunov-gain.* System (2) is *input-to-state stable* (ISS) if it has an ISS Lyapunov function.

It is well known, see Sontag and Wang (1996), that the existence of an ISS Lyapunov function is equivalent to the system being ISS in the following sense:

There exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all L_∞-inputs $u(\cdot)$ it holds that