BOUNDARY CONTROL OF SYSTEMS OF CONSERVATION LAWS : LYAPUNOV STABILITY WITH INTEGRAL ACTIONS

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Abstract: A boundary control law with integral actions is proposed for a generic class of two-by-two homogeneous systems of linear conservation laws. Sufficient conditions on the tuning parameters are stated that guarantee the asymptotic stability of the closed-loop system. The closed-loop stability is analysed with an appropriate Lyapunov function. The control design method is validated with an experimental application to the regulation of water depth and flow rate in a pilot open-channel described by Saint-Venant equations. This hydraulic application shows that the control can be robustly implemented on nonhomogeneous systems of nonlinear conservation laws.

Keywords: Lyapunov stability, Proportional-Integral control, Saint-Venant equations, Systems of conservation laws,

1. INTRODUCTION

In this paper, we are concerned with two-by-two systems of conservation laws that are described by hyperbolic partial differential equations, with one independent time variable $t \in [0, \infty)$ and one independent space variable on a finite interval $x \in [0, L]$. Such systems are used to model many physical situations and engineering problems. A famous example is that of Saint-Venant (or shallow water) equations which describe the flow of water in irrigation channels and waterways. This example will be presented in Section 4. Other typical examples include gas and fluid transportation networks, packed bed and plug-flow reactors, drawing processes in glass and polymer industries, road traffic etc. For such systems, the boundary control problem that we address is the problem of designing feedback control actions at the boundaries (i.e. at $x = 0$ and $x = L$) in order to ensure that the smooth solution of the Cauchy problem converges to a desired steady-state.

The present paper is in the direct continuation of our previous paper (Coron et al. (2007)) where a static feedback control law was presented and the closed-loop stability analysed with an appropriate Lyapunov function. But obviously, a static control law may be subject to steady-state regulation errors in case of constant disturbances or model inaccuracies. In the present paper, we show how additional integral actions can be introduced in the control law in order to cancel the static errors and how the Lyapunov function can be modified in order to prove the asymptotic stability of the closed-loop system. The statement of the control law and the Lyapunov stability analysis are developed in Sections 2 and 3 for a generic *homogeneous* system of two *linear* conservation laws. In Section 4, we present an experimental validation on a laboratory pilot plant. This hydraulic application clearly shows that the control can be robustly implemented on *nonhomogeneous* systems of *nonlinear* conservation laws.

2. STATEMENT OF THE CONTROL LAW

We consider the class of two-by-two systems of linear conservation laws of the general form:

$$\partial_t h(t,x) + \partial_x q(t,x) = 0 \tag{1}$$

$$\partial_t q(t,x) + \alpha\beta\partial_x h(t,x) + (\alpha-\beta)\partial_x q(t,x) = 0 \tag{2}$$

where

- t and x are the two independent variables : a time variable $t \in [0,+\infty)$ and a space variable $x \in [0,L]$ on a finite interval;
- $(h,q) : [0,+\infty) \times [0,L] \to \mathbb{R}^2$ is the vector of the two dependent variables (i.e. $h(t,x)$ and $q(t,x)$ are the two states of the system);
- α and β are two real positive constants:

$$\alpha > \beta > 0.$$

The first equation (1) can be interpreted as a mass conservation law with h the density and q the flux. The second equation can then be interpreted as a momentum conservation law. As usual in control design, the model (1)-(2) must be viewed as a linear approximation of the system dynamics around a steady-state. This will be illustrated with the application of Section 4.

We are concerned with the solutions of the Cauchy problem for system (1)-(2) over $[0,+\infty) \times [0,L]$ under an initial condition

$$h(0,x), q(0,x) \qquad x \in [0,L].$$

Furthermore, it is assumed that the system is subject to physical boundary conditions that can be assigned by an external operator and are written in the following general abstract form:

$$g_0(h(t,0), q(t,0), u_0(t)) = 0 \quad t \in [0,+\infty) \tag{3a}$$

$$g_L(q(t,L), h(t,L), u_L(t)) = 0 \quad t \in [0,+\infty) \tag{3b}$$

with $g_0, g_L : \mathbb{R}^3 \to \mathbb{R}$. The functions u_0, u_L: $[0,+\infty) \to \mathbb{R}$ represent the boundary control actions that can be manipulated by the operator. A concrete illustration of such boundary conditions will be given in Section 4.

In order to define the feedback control laws, it is convenient to introduce the *Riemann coordinates* (see e.g. Lax (1973)) defined by the following change of coordinates:

$$a(t,x) = q(t,x) + \beta h(t,x) \tag{4a}$$

$$b(t,x) = q(t,x) - \alpha h(t,x). \tag{4b}$$

With these coordinates, the system (1)-(2) is rewritten under the following diagonal form:

$$\partial_t a(t,x) + \alpha\partial_x a(t,x) = 0 \tag{5a}$$

$$\partial_t b(t,x) - \beta\partial_x b(t,x) = 0. \tag{5b}$$

The change of coordinates (4) is inverted as follows:

$$h(t,x) = \frac{a(t,x) - b(t,x)}{\alpha+\beta} \tag{6a}$$

$$q(t,x) = \frac{\alpha a(t,x) + \beta b(t,x)}{\alpha+\beta}. \tag{6b}$$

In the Riemann coordinates, the control problem can be restated as the problem of designing the control laws in such a way that the solutions $a(t,x)$ and $b(t,x)$ converge to zero. We shall show that this problem can be solved by selecting the boundary control laws $u_0(t)$ and $u_L(t)$ such that the Riemann coordinates satisfy *linear* boundary conditions of the following form:

$$a(t,0) + k_0 b(t,0) + m_0 y_0(t) = 0 \tag{7a}$$

$$b(t,L) + k_L a(t,L) + m_L y_L(t) = 0 \tag{7b}$$

where k_0, k_L, m_0, m_L are constant tuning parameters while y_0 and y_L are integrals of the flow $q(t,0)$ and the density $h(t,L)$ respectively:

$$y_0(t) = \int_0^t q(s,0)ds = \int_0^t \frac{\alpha a(s,0) + \beta b(s,0)}{\alpha+\beta} ds \tag{8a}$$

$$y_L(t) = \int_0^t h(s,L)ds = \int_0^t \frac{a(s,L) - b(s,L)}{\alpha+\beta} ds. \tag{8b}$$

Remarks

1) Conditions (7) give only an implicit definition of the control laws. The derivation of explicit expressions obviously requires an explicit knowledge of the functions g_0 and g_L in (3). In the special case where the boundary conditions (3) are linear, u_0 and u_L reduce to standard Proportional-Integral (PI) control laws. This point will be further illustrated in Section 4.

2) In our previous paper (Coron et al. (2007)), we have dealt with the special case without integral actions, i.e. m_0=m_L=0 in (7). We have shown