PORT-BASED ENERGY BALANCE ON COMPACT MANIFOLDS

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Abstract: The purpose of this paper is to introduce a novel energy balance structure based on a port-representation for closed manifolds with potential in terms of the Morse theory. The Morse theory states that the local structure around non-degenerate critical points of Morse functions on manifolds reflects the global structure of the whole manifold. The energy balance is then connected to the topological properties of the manifolds and is defined on a non-uniform boundary characterized by the dimensions of submanifolds with outflows. First, we discuss the non-uniform boundary in relation to a Morse-Smale gradient flow. Next, the dual pair of energy variables in the context of ports is defined by using the Poincaré duality theorem. Finally, we present two specific energy balances on the compact manifolds are presented.

Keywords: Algebraic approaches, Geometric approaches, Boundary detection, Singular points, Multiport networks, Nonlinear systems

1. INTRODUCTION

A port-based representation (van der Schaft, 2000) is known as a unified approach for modeling control systems with a symmetry based on an energy conservation law. The advantages of systems using these representations are as follows: (i) The system is based on a passivity that is familiar to engineers. (ii) The intuitive method, a damping injection can be applied to their input/output variable pairs called the ports for stabilization. (iii) We can easily connect each system with ports while preserving the conservative property. This type of symmetry, which is a time invariance of Hamiltonian, gives us the energy balance equation having a clear physical means. The energy preserving property can be expressed by a Dirac structure (Courant, 1990). On the other hand, the port-representation extended to distributed parameter systems is called a distributed port-Hamiltonian system (van der Schaft and Maschke, 2002). The distributed port-Hamiltonian system is defined on a manifold with a boundary. The energy changes of the internal domain can be integrated by the Stokes's theorem; the fundamental structure is called a Stokes-Dirac structure. The energy balance on the boundary can be used for a boundary observer of the internal energy change from a practical viewpoint.

In this paper, we present two energy balances for a closed manifold M in terms of the Morse theory (Milnor, 1963; Matsumoto, 2002; Banyaga and Hurtubise, 2004). The energy balances are defined on particular boundaries in the context of the distributed port-representation. The original concept of the Morse theory is to investigate the global property of the manifold from the local structures around non-degenerate critical points. Moreover, the global topology of the whole manifold determines some of the permissible local structures around the critical points. The new energy balances are induced by the topological properties of the manifolds; the corresponding boundaries are classified by a Morse index of the critical point. We presented the formal structure that expresses the relation between the differential forms defined around the non-degenerate critical points called a Morse-Smale-Dirac structure (Nishida *et al.*, 2006). The goal of this study is to define the energy variables from an energy flow generated by a Morse function on the manifold. Our results give a geometrical interpretation to the Morse-Smale-Dirac structure in contrast with its algebraic representation.

Let $f\colon M \to \mathbb{R}$ be a Morse function. A smooth gradient flow $-\nabla f$ of f on M has the zero map $df_p = 0$ at p if M is compact, where $df_p\colon T_pM \to \mathbb{R}$. The point p is called a critical point. Both attracting and repelling flows exist toward p as a gradient flow $-\nabla f$ around p. We define a dimension of an unstable manifold of a critical point as a Morse index $\lambda = \mathrm{ind}(p)$. Tracking the orbit $\{\cdots \to p^{\lambda+1} \to p^{\lambda} \to p^{\lambda-1} \to \cdots\}$ connecting such critical points as p^λ of index λ, we can obtain a notable region around p^λ for the energy balances of the following two boundaries. One of them is in the small domain around one of the critical points p^λ of f. The local boundary consists of in/out flows according to the index. The flows are a stationary phenomenon expressed as an invertible 1-parameter family on the manifolds (e.g. the current at the electrical circuits node, the stream in the junction of some rivers, the data transferring through the hub of communication networks, the wind through a room that has windows, and the traveling acoustic wave enclosed with a multi-way pipe). The other is a layer that contains all the critical points $p_1^\lambda, \cdots, p_n^\lambda$ of the same index λ. The global boundary corresponds to the level sets crossing M transversally.

2. MATHEMATICAL BACKGROUND

2.1 Differential forms and de Rham cohomology

Let M be an n-dimensional C^∞ manifold. We denote the vector space of all k-forms on M by $\Omega^k(M)$, where k is an integer such that $0 \le k \le n$. Let $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ denote an *exterior differential operator*. A k-form $\omega \in \Omega^k(M)$ is called a *closed form* if $d\omega = 0$, and an *exact form* if a $(k-1)$-form η such that $\omega = d\eta$ exists. Let us denote the set of all closed k-forms on M by $Z^k(M)$ and the set of all exact k-forms by $B^k(M)$. The quotient space $H^k_{DR}(M) = Z^k(M)/B^k(M)$ is called the *k-dimensional de Rham cohomology* of M.

2.2 Morse theory

Let M be an n-dimensional closed manifold and $f\colon M \to \mathbb{R}$ be a smooth function. If a differential $df_p\colon T_pM \to \mathbb{R}$ is a zero map, p is a *critical point* of f. The f is called a *Morse function*, if every critical point p is a non-degenerate $\det Hf_p \neq 0$, where $Hf_p = \partial^2 f(p)/(\partial x_i \partial x_j)$ is a Hessian. The number of negative eigenvalues of Hf_p is called an *index* of p. From the Morse lemma, we can take a suitable local coordinate $(x_1, \cdots, x_n)$ in a neighborhood of p of index λ so that the function f has a standard form given by

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2 . \quad (1)$$

Let us consider a negative gradient flow $\dot{x} = -\nabla f(x)$ on M. Let ϕ^t be the generated invertible 1-parameter family of $-\nabla f$. Now, we interpret the flow as a differentiable manifold itself. Let S be a set of critical points of M. For all $p \in S$ of f,

$$W^s(p) = \{x \in M; \lim_{t\to+\infty} \phi^t(x) = p\}, \quad (2)$$

$$W^u(p) = \{x \in M; \lim_{t\to-\infty} \phi^t(x) = p\} \quad (3)$$

are a *stable manifold* and an *unstable manifold*, respectively. All points on M except for the critical points are on one integral curve. Every integral curve starting from $p \in S$ of index λ arrives at the critical points of index $\lambda - 1$ or less. For $p \in S$ of index λ, $W^s(p)$ is an $(n-\lambda)$-dimensional submanifold of M , $W^u(p)$ is an λ-dimensional submanifold of M, and $W^s(p) \cap W^u(p) = \{p\}$. Then, $W^u(p) \cap W^u(q) = \emptyset$ for $q, p \in S$ such that $p \neq q$ and $\bigcup_p W^u(p) = M$. Therefore, $\{W^u(p)\}$ gives a decomposition of M with a cell, which is the homeomorphic topological space with respect to $\mathbb{R}^\lambda$. Let us consider an intersection $I(p,q) = W^u(p) \cap W^s(q)$ for any p, $q \in S$ such that $p \neq q$. Generally, $I(p,q)$ is a complicated-shaped subset of M. Then, we assume that $W^u(p)$ and $W^s(q)$ intersect transversely, which is a *Morse-Smale condition*, that is, $\dim(I(p,q)) = \mathrm{ind}(p) - \mathrm{ind}(q)$. The corresponding Morse function is called a *Morse-Smale function*. Under this condition, $I(p,q)$ is a submanifold of M. In particular, if $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, then $W^u(p) \cap W^s(q) = \bigcup_i I_i(p,q)$ and $I_i(p,q) \cap I_j(p,q) = \emptyset$ for $i \neq j$, where each $I_i(p,q) \simeq \mathbb{R}$ is an image of integral curves.

It is known that the homology groups of *CW-complexes* can be introduced with an orientation on each cell in E (Matsumoto, 2002; Morita, 2001; Madsen and Tornehave, 1997). The CW-complex can be associated to a Morse function. Now,