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Concise presentation of the Coulomb electrostatic potential of a uniformly charged cube

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ABSTRACT

We use a novel method to calculate in closed form the Coulomb electrostatic potential created by a uniformly charged cube at an arbitrary point in space. We apply a suitable transformation of variables that allows us to obtain a simple presentation of the electrostatic potential in one-dimensional integral form. The final concise closed form expression of the Coulomb electrostatic potential of the uniformly charged cube is obtained after completing the calculation of the resulting one-dimensional integrals. Such integrals consist of combinations of products of error functions and power functions that can be solved exactly despite their intimidating appearance. The exact analytic formula for the Coulomb electrostatic potential that we derive reflects the symmetry of the cube and is easy to implement. We illustrate its use by calculating the exact values of the electrostatic potential at some points of symmetry such as the center of cube, center of face of cube, center of edge of cube and corner of cube.

Introduction

Calculation of the electrostatic potential created by charged bodies is very important to many fields [1–3]. Obviously, if the charged body under consideration has arbitrary shape and if the distribution of charge is arbitrary, then no exact analytical results are available. However, if the charged body has regular features/symmetries and is uniformly charged, then one may have a chance to calculate its electrostatic potential in analytic form. This is the reason why regular bodies have always drawn a huge interest over long time periods for their importance to various models in physics and mathematics. In particular, uniformly charged bodies with spherical symmetry (spheres, spherical shells, etc) or cylindrical symmetry (cylinders, disks, etc) allow one to obtain exact expressions [4,5] for the electrostatic potential in some cases quite trivially (for instance, the case of spheres). On the other hand, uniformly charged bodies with nospherical shape are notoriusly more challenging. Interest on them stems from the fact that geometric figures with nospherical shape (for instance, polyhedra where the cube is the simplest of them) can be used as models for irregular bodies. Therefore, it is not surprising to discover that some of the earlier work found in the literature dealt with cubes. In particular, astrophysical studies of gravitational forces dealt with the calculation of the gravitational potential created by a uniformly filled (homogeneous) polyhedra (with cube as a special case) [6,7]. To our knowledge, the gravitational potential of a uniformly filled parallelopiped (cube) was apparently first calculated explicitly by MacMillan [8]. Calculations are difficult and the work represents a masterful use of direct integration techniques. The final analytic expression obtained in closed form contains a very large number of terms and is quite long [8].

The key objective of this work is to calculate the Coulomb electrostatic potential created by a uniformly charged cube at some arbitrary point in space by adopting a different approach that simplifies the calculations and leads to a much more concise presentation of the final result. The method used differs from the direct integration approach implemented by other authors [8–11]. Another benefit of this method is that it allows us to obtain a very simple presentation of the electrostatic potential as a one-dimensional integral function. Explicit calculation of the resulting one-dimensional integrals leads us to the final closed form exact result, a compact sum of only eight terms.

The model under consideration consists of a cube with length, L uniformly filled with some positive charge, Q. We want to calculate exactly the Coulomb electrostatic potential created by such a uniformly charged cube at some arbitrary point, $\vec{r} = (x, y, z)$ in three dimensions (3D). We write the expression for the electrostatic potential at any arbitrary point in 3D space as:

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$$V(x,y,z) = k_e\rho_0 \int_D d^3r' \frac{1}{|\vec{r} - \vec{r}'|}, \quad (1)$$

where k_e is Coulomb's electric constant, $\rho_0 = Q/L^3$ is the uniform 3D charge density of the cube, $\vec{r}'$ and $\vec{r}$ are 3D position vectors and D is the cubic domain of integration. We choose the origin of a Cartesian system of coordinates at the center of cube and the axes parallel to its edges, thus:

$$D: -\frac{L}{2} \le x', y', z' \le +\frac{L}{2}. \quad (2)$$

Our approach hinges upon a suitable transformation of $1/|\vec{r} - \vec{r}'|$ as a function of auxiliary variables, a transformation that eventually leads us to a simpler expression for the electrostatic potential. In the present case, the method leads to a very concise presentation of the potential, firstly, in one-dimensional integral form and, secondly, in closed analytic form, once the integrals are calculated. The integral presentation of the Coulomb electrostatic potential is convenient for numerical calculations since the integrand function is easy to handle numerically. Even though the calculation of the resulting one-dimensional integrals is challenging, we were able to calculate them exactly in analytic form. Differently from the direct integration approach sketched in Appendix A, our method leads to certain type of integrals that are not commonly found in textbooks. For this reason, we deem it important to provide a detailed description of several integration formulae that we derive. These mathematical results can be found in Appendix B, Appendix C, Appendix D, Appendix E and Appendix F. These formulae allow us to complete the calculations rather easily, thus, they are essential to this work. Only after having obtained all the required analytical formulae for various non-standard integrals appearing during the solution process, we proceed to obtain an explicit analytical expression. Since the calculation of the electrostatic potential at points of symmetry is important, we also provide the exact values of the electrostatic potential at the center of cube, center of face of cube, center of edge of cube and corner of cube.

In Method we explain the general solution method. In Electrostatic potential at the center of cube, Electrostatic potential at the center of face of cube, Electrostatic potential at the center of edge of cube and Electrostatic potential at the corner of cube we list the results for the value of the Coulomb electrostatic potential created by a uniformly charged cube at given special points of symmetry. The general closed form result for the Coulomb electrostatic potential created by an arbitrarily charged cube in the surrounding space is given in Electrostatic potential at an arbitrary point. A brief summary of the work is given in Conclusions.

Method

As outlined in Appendix A, calculation of $V(x,y,z)$ in Eq. (1) by direct integration does not appear to be extremely challenging at the beginning. However, as shown by many authors [8–11], this approach grows in complexity very quickly and, at the end, one has to deal with long cumbersome mathematical expressions. As a result, one is always tempted to find alternative routes to solve the problem in a quicker and more transparent way. In this work we solve the problem by adopting a different approach that does not start with direct integration. To this effect, we write the quantity $1/|\vec{r} - \vec{r}'|$ in Eq. (1) as an integral function of a new auxiliary variable:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{2}{\sqrt{\pi}} \int_0^\infty du\, e^{-u^2(\vec{r} - \vec{r}')^2}. \quad (3)$$

The anticipation is that, this way, the integration over the variables x', y' and z' that appear in the expression in Eq. (1) will be carried out without much difficulty. Note that:

$$(\vec{r} - \vec{r}')^2 = (x - x')^2 + (y - y')^2 + (z - z')^2. \quad (4)$$

By substituting the result from Eq. (3) into Eq. (1), one has:

$$V(x,y,z) = k_e\rho_0 \frac{2}{\sqrt{\pi}} \int_0^\infty du \int_{-\frac{L}{2}}^{+\frac{L}{2}} dx' e^{-u^2(x-x')^2} \times \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' e^{-u^2(y-y')^2} \int_{-\frac{L}{2}}^{+\frac{L}{2}} dz' e^{-u^2(z-z')^2}. \quad (5)$$

The integrals with respect to x', y' and z' are easy to calculate [12,13]. In addition, we can simplify the expressions if we introduce dimensionless variables: $X = x/L$, $Y = y/L$, $Z = z/L$ and $X' = x'/L$, $Y' = y'/L$, $Z' = z'/L$. With the change of variables, we have: $-1/2 \le X', Y', Z' \le +1/2$. In terms of the new variables, $du\, dx'\, dy'\, dz' = L^2\, dt\, dX'\, dY'\, dZ'$, where $t = u\,L$ is another dummy variable. Straightforward calculations lead us to the following expression for the Coulomb electrostatic potential created by a uniformly charged cube:

$$V(X,Y,Z) = \frac{k_e Q}{L} \frac{2}{\sqrt{\pi}} \int_0^\infty dt\, f(t,X) f(t,Y) f(t,Z), \quad (6)$$

where $f(t,X)$, $f(t,Y)$ and $f(t,Z)$ are auxilary functions. Specifically:

$$f(t,X) = \frac{\sqrt{\pi}}{2t} \left\{ erf\left[t\left(\frac{1}{2} - X\right)\right] + erf\left[t\left(\frac{1}{2} + X\right)\right] \right\}, \quad (7)$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}, \quad (8)$$

is an error function (See Pg. 297 of Ref. [14]). Note that $f(t,Y)$ and $f(t,Z)$ are obtained from Eq. (7) by, respectively, replacing X with Y and X with Z. Note that the form of the auxiliary functions reflects the symmetry of the problem under consideration. For instance, one can see that:

$$f(t,-X) = f(t,X). \quad (9)$$

Obviously, the result in Eq. (9) mirrors the fact that $V(-X,Y,Z) = V(X,Y,Z)$ which is easy to deduce from the cubic symmetry. The integral presentation in Eq. (6) is not only simple, but also very convenient for numerical calculations since the auxiliary functions in the integrand are smooth and do not have singularities. For instance, it is easy to verify that, for a given finite value of X:

$$\lim_{t\to 0} f(t,X) = 1; \lim_{t\to\infty} f(t,X) = 0. \quad (10)$$

The usefulness of the result in Eq. (6) is self-evident if, for example, one wants to calculate the Coulomb self-energy (E_S) of a uniformly charged cube:

$$E_S = \frac{k_e\rho_0^2}{2} \int_D d^3r \int_D d^3r' \frac{1}{|\vec{r} - \vec{r}'|} = \frac{\rho_0}{2} \int_D d^3r V(\vec{r}). \quad (11)$$

It is straightforward to prove that: