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Analytical expression for the electric field enhancement between two closely-spaced conducting spheres

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ABSTRACT

An external electric field applied to two conducting spheres in close approach is enhanced (by charge separation on the spheres) in the region between the spheres. For spheres of equal size, this enhancement is a universal function of the ratio of the separation of the spheres to their radius, and increases without limit as this ratio decreases. We calculate the enhancement factor analytically for perfectly conducting spheres, providing a simple formula valid when the spheres are close together, that is when the enhancement is large and the known series solution is difficult to evaluate. The same methods allow us to find the close-approach forms of the longitudinal and transverse polarizabilities of the two-sphere system.

1. Introduction

The stimulus for this work comes from the very large electric field enhancement factors that have been achieved in surface enhanced Raman spectroscopy, enabling the detection of Raman signals from single molecules [1,2]. The problem that we discuss here, that of enhancement of the external electric field by two metal spheres, has been treated both for perfectly conducting spheres [3–9], and with realistic values of the metal dielectric function at optical frequencies [10–20]. Even in the perfectly conducting case, the solution requires the computation of infinite series, which converge more and more slowly as the spheres come closer together. In the case of dielectric spheres, an infinite set of equations for the expansion coefficients must be truncated and then solved, before the series are summed.

Here we return to the problem of two perfectly conducting spheres, and provide a simple formula which becomes more accurate as the physically interesting limit of nearly touching spheres is approached. This formula allows rapid back-of-the-envelope estimates of the field enhancement for a given ratio of the sphere separation s to radius r of the spheres. The field enhancement factor to be derived is

$$f = E_{\mathrm{ave}}/E_0 \approx \frac{\pi^2}{3}\frac{r}{s}\left[\frac{1}{2}\ln\frac{r}{s} + \ln 2 + \gamma\right]^{-1}$$

E_0 is the external electric field, E_{ave} the average field in the gap between the spheres, along the line of centres. For $r/s = 1000$ ($r = 1\ \mu$m, $s = 1$ nm, for example), this formula gives 696.38 for the field enhancement ratio, accurate to 1 part in 10,000.

With the same techniques, we also derive the near-approach analytic forms of the longitudinal and transverse polarizabilities of the two-sphere system, and show that for sphere separation s much smaller than the sphere radius r these are

$$\alpha_\ell \approx r^3\left\{4\zeta(3) - \frac{\pi^4}{18}\left[\frac{1}{2}\ln\frac{r}{s} + \ln 2 + \gamma\right]^{-1}\right\}$$

$$\alpha_t \approx r^3\left\{\frac{3}{2}\zeta(3) + \left[\frac{3}{4}\zeta(3) - \ln 2\right]\frac{s}{r}\right\}$$

Here $\gamma \approx 0.5772$ is Euler's constant and $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.202$.

The results are restricted to the electrostatic problem with perfectly conducting spheres, which is adequate provided the spheres are small compared to the wavelength of the incident light, and provided the real part of the dielectric function of the spheres is numerically large, so that electric fields are normal to the conductor surfaces. The latter proviso holds quite well in the far infrared, but is not so good in the visible.

The electrostatic problem is discussed in Maxwell's treatise [3], and a complete solution was given by Jeffery [4] nearly a century ago, using bispherical coordinates. His solution is in terms of two infinite series, which converge rapidly provided the separation of

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the spheres is not small compared to their radius. However, it is precisely at very small separations that the field enhancement is large, and we provide analytical formulae which apply in this limit. More recent work [5–9] (none quoting Jeffery, incidentally) does not consider this limit.

2. The solution in bispherical coordinates

Fig. 1 shows the orthogonally intersecting circles which form the bispherical coordinate system (u,v), defined by

$$u + iv = \ln\frac{\rho + i(z+b)}{\rho + i(z-b)} \tag{1}$$

Here $\rho^2 = x^2 + y^2$, and the surfaces $u =$ constant are spheres whose centres lie on the axis of symmetry, the z-axis. The length b determines the scale of the diagram. In terms of the bispherical coordinates (u,v) the radial and axial coordinates are given by

$$\rho = b\frac{\sin v}{\cosh u - \cos v}, \quad z = b\frac{\sinh u}{\cosh u - \cos v} \tag{2}$$

If we set $u = u_0$ and eliminate v, we obtain the circles

$$\rho^2 + (z - b\coth u_0)^2 = b^2 \operatorname{cosech}^2 u_0 \tag{3}$$

The orthogonal system of circles results when we set $v = v_0$ and eliminate u:

$$(\rho - b\cot v_0)^2 + z^2 = b^2 \operatorname{cosec}^2 v_0 \tag{4}$$

All of the $v =$ constant circles pass through the points $\rho = 0$, $z = \pm b$. From (3), the surfaces $u = \pm u_0$ are spheres with radii r_0 and centres at $\pm z_0$, where

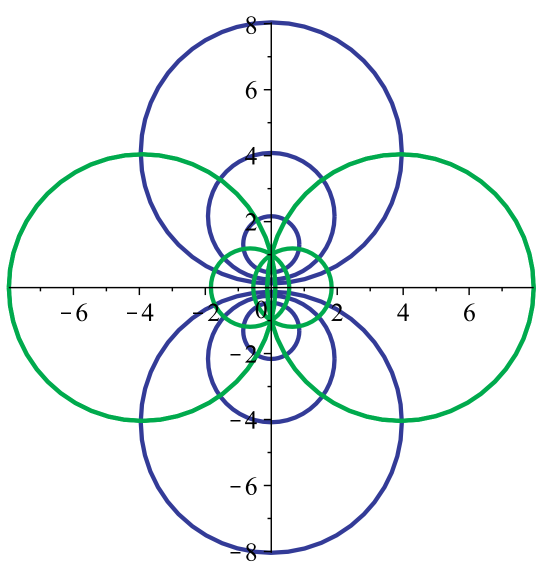

Fig. 1. The bispherical coordinate system. The z-axis is vertical, x and y axes are horizontal. The scale length b has been set equal to unity. The values $u = 1, 1/2, 1/4$ give the circles centred on the upper z-axis, and the same negative values give the circles on the lower z-axis. The inner circles are $u = 1, -1$. As u decreases the circles get larger, and closer together. The three-dimensional picture is obtained by rotation of the figure about the z-axis. In the problem considered here, the $u =$ constant circles (rotated about the z-axis) represent the spherical conductors. The orthogonal system of circles then gives the (toroidal) field lines in the case of oppositely charged conductors in zero external field. It is illustrated by the values $v = 1, 1/2, 1/4$.

$$r_0 = \frac{b}{\sinh u_0}, \quad z_0 = b\,\frac{\cosh u_0}{\sinh u_0} \tag{5}$$

The ratio of the distance between the centres to the diameter of the spheres is thus $\cosh u_0$. We are interested in close approach, when this ratio is close to unity, and thus in small u_0.

Let $V(\rho,z)$ be the electrostatic potential for the two-sphere problem. Jeffery [4] shows that

$$V_n(\rho,z) = (\cosh u - \cos v)^{1/2} \sinh\left(n + \frac{1}{2}\right)u\, P_n(\cos v) \tag{6}$$

solves Laplace's equation. (He also considers the more general case with azimuthal dependence, which we do not need here.) Suppose the external electric field is downwards in Fig. 1, and the spheres $u = \pm u_0$ are uncharged and at potentials $\pm V_0$. Then the potential on and outside of the two spheres is given by [5,6]

$$V(\rho,z) = E_0 z + (\cosh u - \cos v)^{1/2} \times \sum_{n=0}^{\infty} A_n \sinh\left(n + \frac{1}{2}\right)u P_n(\cos v) \tag{7}$$

where

$$A_n = 2^{3/2}[V_0 - (2n+1)E_0 b]\left[e^{(2n+1)u_0} - 1\right]^{-1} \tag{8}$$

and

$$V_0 = E_0 b\,\frac{N(u_0)}{D(u_0)} \tag{9}$$

The numerator N and denominator D in the last expression are the infinite sums

$$N(u) \equiv \sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)u} - 1}, \quad D(u) \equiv \sum_{n=0}^{\infty} \frac{1}{e^{(2n+1)u} - 1} \tag{10}$$

The external electric field E_0 separates charges on the (neutral) spheres, placing opposing charges on the nearby parts of the spheres, thus increasing the local electric field. When the spheres are close, the increase can be very large. The enhancement factor f is given by

$$f = \frac{N(u_0)}{D(u_0)} \coth\frac{u_0}{2} \tag{11}$$

This is the ratio of the average field along the z-axis in the gap of length s_0 between the spheres, to the external field E_0. The separation distance (the distance of closest approach) is

$$s_0 = 2z_0 - 2r_0 = 2b \tanh\frac{u_0}{2} \tag{12}$$

and the voltage difference between the spheres is $2V_0$, so $f = E_{ave}/E_0 = (2V_0/s_0)/E_0$, which gives (11) on substitution from (9) and (12).

The maximum local value of E is larger than E_{ave}, and occurs on the z-axis, at $z = \pm s_0/2$. When the spheres are close together relative to their radii, E_{max} tends to E_{ave}. When the spheres are far apart compared to their radii, E_{max} tends to $3E_{ave}$, since then the average field tends to E_0, and the maximum field to $3E_0$ (the value at the poles of the spheres). Thus E_{max}/E_{ave} varies from three to unity. The full expression [7] for E_{max} is considerably more complicated than that for E_{ave}. Since E_{max} tends to E_{ave} in the physically interesting large-enhancement limit, we shall use the simpler enhancement ratio $f = E_{ave}/E_0$.