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Original

Analytical and *Numerical funicular analysis* by means of the *Parametric Force Density Method*

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Abstract

The *funicular* concept has often been used in different stages of structural analysis and design. This paper presents two new methods: *Analytical*, A-*FDM*, and *Numerical method*, N-*FDM*, based on a parametric application of the original *Force Density Method* (*FDM*). This is an especially useful way of visualizing a set of solutions and optimizing, i.e. selecting one specific *funicular* related to a set of constraints. Two structural algorithms are implemented iteratively with *Maple*® in real time, and output is also linked to *AutoCAD*®. *Maple*® facilitates control of geometrical constraints, while *AutoCAD*® helps to show all parameterized data. Because of their practical interest, special emphasis is placed on masonry structures using a *Limit Analysis approach* and preliminary design. Examples of the application of both methods are depicted.

Keywords: *Form-finding*; Parametric; *Force Density Method*; *Funicular analysis*; Masonry structures; Preliminary design

1. Introduction

1.1. Form-finding versus the Force Density Method

Most unstressed 2D and 3D tensile bar structures are kinematically indeterminate. As a result, their final equilibrium configuration geometry, i.e. the position of the nodes, is a priori unknown. The search for an initial shape compatible with a set of loads and constraints is termed *form-finding*.

A tensile structure can be seen as a materialization of a 3D *funicular*. This is also the case for masonry structures when a *Limit Analysis approach* is used, as the problem here is also based on the *funicular* concept. The link between *form-finding methods* and *funicular analysis* is therefore straightforward.

The *Force Density Method, FDM* (Linkwitz & Schek, 1971; Schek, 1974) was developed in the 1970s as a *form-finding* procedure for cable tensile structures (Grundig, Moncrieff, Singer, & Ströbel, 2000).

FDM was selected for this research due to four main considerations: (1) It manages equilibrium equations in a totally direct way, and is therefore especially suited for a *funicular* solution; (2) equilibrium equations are linearized, which simplifies the numerical process, even though an iterative analysis is usually needed; (3) no pre-sizing is required for this method; this is a crucial question for many approaches and particularly for the two new applications addressed; and (4) the three equilibrium equations are *uncoupled*, an important property that will be exploited here.

1.2. Funicular analysis versus masonry structures

Funicular analysis refers to the use of a 2D or 3D funicular as an analytical tool at any stage of the analytical process. Additional assumptions would also make it a design tool, as in the case of masonry structures *Limit Analysis*. The *funicular* concept is not restricted to linear elements, e.g. cable structures, but could also be applied to surface elements, e.g. for creating membranes.

This paper will describe a wire frame model, either linked with linear elements or representing membrane discretization, with special focus on the case where there is only tension or internal compression force; although some procedures are valid

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for both tension and compression if proper constraints are considered.

The first application of *Limit Analysis theory* (Kooharian, 1952) for the analysis and design of masonry structures has being notably expanded and consolidated (Heyman, 1966, 1969).

The assumptions inside this frame are: (i) Constitutive equations are rigid-plastic, with no tensile strength but infinite compressive strength, (ii). Friction between the voussoirs is sufficient to prevent failure due to sliding between them; (iii). Stability is only considered inside a rigid multi-body model and according to the first assumption (i).

A high span or great depth of the mortar between voussoirs would make the first assumption impossible, (i) Assumption (ii) can be checked a posteriori. The validity of assumptions (i) and (iii) run quite parallel. Nevertheless, in the majority of cases, the said assumptions can be applied, and then *funicular analysis* is really simple: the structure is safe (*Lower Bound Theorem* or *Safe Theorem*) if at least one thrust-line, i.e. a *funicular*, can be traced inside the geometrical boundaries of the structure (Poleni, 1748). This is in fact quite an old supposition, but once it has been included within the framework of *Limit Analysis* its level of reliability becomes clear. The use of *form-finding methods* for *funicular analysis* is therefore totally justified.

The question of adding constraints for selecting a particular *funicular* has been approached in different ways. One of the first was to use linear programming (Livesley, 1978). In the *Force Network Approach, FNA* (O'Dwyer, 1999) the equilibrium path is fixed in one plane, in this case in the horizontal one, i.e. the projection of the 3D funicular in this plane therefore the thrust is fixed. Afterwards, the ordinates of the funicular target are obtained by linear programming. This method limits its application to the case where loads are perpendicular to the plane where the equilibrium path is fixed (usually the horizontal one), which is its most important drawback.

The idea of fixing the 3D funicular projection into a plane together with thrust in the corresponding directions had been proposed for cable tensile structures, and is known as the *Grid Method*, *GM* (Siev & Eidelman, 1964). The condition of vertical equilibrium makes it possible to obtain coordinates perpendicular to the grid, giving rise to a system of linear equations. *GM* was also limited to the case of load perpendicular to the grid. A similar approach was used in fixing the horizontal path of the funicular in a grid together with thrust. Equilibrium is resolved iteratively node by node (Berger, 1996).

FDM is especially suited for fixing the *funicular* path in one plane, as the equilibrium equations in three perpendicular directions are *uncoupled*. As was pointed out above, this property is one of the main advantages of the method.

Thrust Network Analysis, *TNA* (Block, Ciblac, & Ochsendorf, 2006; Block & Ochsendorf, 2007; Block, 2009) is strongly connected with *FNA*, but adds parallel handling of the reciprocal or dual figure to horizontal projection, i.e. the force diagram, and the use of the *FDM*.

The *Analytical*, A-*FDM*, and the *Numerical method*, N-*FDM*, are both described in this article. They are based on parametric application of the original *FDM* for obtaining *funicular* solutions, and were independently proposed by the authors (Cercadillo-García & Fernández-Cabo, 2010; Cercadillo-García, 2014).

1.3. Funicular analysis versus preliminary design

Preliminary design refers to the application of a 2D or 3D *funicular* for selecting the initial shape of a structure, assuming that a *funicular* shape leads to high structural efficiency.

The use of physical models to support preliminary design has been present throughout the history of construction. Hanging models have been used to trace the *funicular*. The well-known case of *Antoni Gaudí* may constitute the highest expression. In the 1960s and 1970s physical models were replaced by computer models.

Tensile structures needed computer models, and *funicular analysis* is now consolidated as an independent area. Together with other new architectural lines such as using *free* (i.e. *organic*) *forms*. Computational improvements are promoting and challenging this working line (Kilian & Ochsendorf, 2005).

1.4. Funicular analysis versus the parametric method

Parametric refers in part to the parametric capability of tools used in symbolic computation in e.g. *Maple*®; but it mainly describes to the nature of the proposed method, such as searching for a specific *funicular*, which is parameterized in terms of independent variables.

This paper presents a new method, the *Parametric Force Density Method*, for tracing a selected 2D or 3D *funicular* (Cercadillo-García, 2014). This method is developed in different ways: *Analytical*, A-*FDM*, and *Numerical*, N-*FDM*, extensions of *FDM*. The application of the methods to the fields of masonry structure and preliminary design are specifically addressed.

The mathematical software *Maple*® is used to implement structural algorithms, and its capability to work symbolically is especially important. *AutoCAD*® is used as a graphical and geometrical tool. *Maple*® results are exported to *AutoCAD*® compatible files, and both environments are linked in real time.

2. Method

2.1. Original FDM

FDM states the problem for a pin-jointed structure of straight bars. Let m be the number of bars, n the number of total nodes of the structure, n_f the number of free or unconstrained nodes, and n_c the number of fixed or constrained nodes. Load, p_i, is located at the nodes. For the node number i, their Cartesian coordinates are (x_i, y_i, z_i).

The *Branch-node matrix* was known originally as the *Incidence matrix*, $[C]$. Its rows are linked with the branches or bars, ordered from 1 to m, and its columns are linked with the nodes (but dividing the free and constrained nodes, as will be shown). If $i(m)$ is the initial node number of the branch m and $j(m)$ its