Synchronization of Irregular Complex Networks with Chaotic Oscillators: Hamiltonian Systems Approach

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ABSTRACT
Synchronization of multiple chaotic oscillators in Hamiltonian form is numerically studied and is achieved by appealing to complex systems theory [1-5]. The topology that we consider is the irregular coupled network. Two cases are considered: i) chaotic synchronization without master oscillator (where the final collective behaviour is a new chaotic state) and ii) chaotic synchronization with master oscillator (where the final collective behaviour is imposed by the dynamics of the master oscillator to multiple slave oscillators). The Hysteretic and Rössler chaotic oscillators in Hamiltonian form will be used as examples.

Keywords: Synchronization, Complex Networks, Hamiltonian Systems.

RESUMEN
La sincronización de múltiples osciladores caóticos en forma Hamiltoniana es numéricamente estudiada y se logra apelando a la teoría de sistemas complejos [1-5]. La topología que consideramos es la red compleja irregular. Dos casos se consideran: i) sincronización caótica sin oscilador maestro (donde el comportamiento colectivo final de la red compleja es un estado caótico nuevo y ii) sincronización caótica con oscilador maestro (donde el comportamiento colectivo final de la red caótica es impuesto por la dinámica del oscilador maestro a los osciladores esclavos). Los osciladores caóticos de Rössler e Histéresis en forma Hamiltoniana se utilizan como ejemplos.

1. Introduction

Since synchronization discovery, the scientific community has paid special attention to this phenomenon due to its importance in science and technology. In the late years, especially since the appearance of the article by Pecora and Carroll [6], the scientific community has turned their eyes into complex network synchronization. Ever since, lots of ideas have emerged, posing interesting new ways to synchronize diverse dynamical systems [7-8], as well as exciting new findings in science.

Some of the first accounts of synchronization behavior might be referenced to the well-known story of the Bible, when the Jericho walls were destroyed by highly coupled sounds of trumpets and an army marching around the city. Nevertheless, the first research report on synchronization in general is due to Christiaan Huygens [9-10] in the mid 1600's. He suspended two pendulum clocks through a common wooden beam. The tiny vibrations from the clocks synchronize the pendulums' movements giving place to rhythmic swings. Ever since this property has been reported in systems of different nature such as chemical, biological, social, physical and so on, e.g. fire-flies bioluminescence communication [11-12], bamboo mast flowering [13], sound pipes quenching [14] and heavenly bodies movements [15]. In general, we can understand synchronization as the "adjustment of rhythms of oscillating objects due to their weak interaction" [16]. It can be said that synchronization is one of the most widespread phenomena among oscillating dynamical systems.

A prevailing occurrence in nature is the gathering of systems or individuals in common tasks, such large numbers of highly interconnected dynamical systems render a collective behavior completely different to the individually shown by its units. Classic examples run from the nervous system (a network by itself) to human societies, from a grass hopper in the garden to swarms of locust on the fields, and from a tiny personal system such as a smart phone to entire computer networks that drive the lives of modern world.

All of these cases can be characterized as structures made of individual entities, called nodes (with an individual behavior known or not), weak connections among them and, depending on the nature of the system, the existence of a driving force which imposes its dynamics to the network, we call this entity a master node. The nodes' dynamical nature plus the structure of the connections dictate the network's behavior, which can be simple or capriciously complicated. These phenomena can be translated to benefices to the human race (e.g. the heart pace maker), which stresses the importance of the study of this topic nowadays.

The nodes used in this work are chaotic oscillators, the richness of its dynamics due to its random behavior challenges many paradigms of synchronization techniques. This chaotic nature became popular among the scientific community in the mid 1960's thanks to Edward Lorenz [17] who, while trying to forecast weather, discovered a defining feature of chaos, any difference in the initial conditions of certain dynamical systems, no matter how small it might be, has huge effects on its outcome. Its understanding unraveled lots of nature's "code", as Prof. Marcus du Sautoy [18-19] would say. Examples of this can be found in the population growth of some insects [20], weather forecasting [17], economics [21-22], social behavior, movement of artificial satellites [23], chemical reactions, electronic circuits [27], etc.

Complex dynamical networks can be defined as an interconnection of individual dynamical systems interacting in several ways [28]. Such interconnections might have different forms or topologies, every arrangement of units affect the final behavior of the system. In networking, irregular arrays have no formal or regular construction, and pose an evolving research topic, showing that even when nature has its own mathematical order, disorder is most likely to be found.

In this work we study the synchronization of complex networks made by chaotic dynamical nodes (we call these oscillators from now) in irregular arrays and two main arrangements, with and without a master node. Such systems exhibit emergence and synchronization to the master node behavior.

The paper is organized as follows: in Section 2 a brief summary on synchronization in complex dynamical networks is given. For Section 3, an approach to solve networks given in Hamiltonian Generalized form, followed by Section 4 where two examples are given. Finally conclusions are given in Section 5. In Appendix A, the essentials of Hamiltonian Generalized form systems, and its synchronization is included.

2. Synchronization of Complex Networks

A complex dynamical network is defined as a set of interconnected oscillators. An oscillator is the basic element of a network, whose behavior depends on its nature [4]. Consider a dynamical network that is made of N identical linearly and diffusively coupled oscillators, and each oscillator is an n-dimensional system with chaotic behavior. The state equations of the complex network are given by:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j,\ i = 1, 2, ..., N, \tag{1}$$

Where $\dot{x}_i = (x_{i1}, x_{i2}, ..., x_{in})^T \in R^n$, is the state vector of the oscillator i. The constant $c > 0$ is the coupling strength of the complex network. $\Gamma \in R^{n\times n}$ is a constant matrix and it is assumed that $\Gamma = diag(r_1, r_2, ..., r_n)$ is a diagonal matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $j \neq i$. This means that two coupled oscillators are linked through their ith state variables. Coupling matrix $\mathbf{A} = (a_{ij}) \in R^{N\times N}$ represents the coupling configuration of the complex network. If there is a connection between oscillator i and oscillator j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0\ \ (i \neq j)$.

The diagonal elements of matrix are defined as

$$a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji},\ i = 1, 2, ..., N. \tag{2}$$