Common Fixed Points of Expansive Mappings in Generalized Metric Spaces

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ABSTRACT
In this paper, we establish a common fixed point theorem for expansive mappings by using the concept of weak compatibility in the setting of G-metric spaces. This result generalizes the result of Ahmed [2] from 2-metric spaces to G-metric spaces by removing the condition of sequential continuity of the mappings. Further, we generalize and extend the theorem of Şahin and Telci [20] to G-metric spaces and thereby extending the theorem of Wang et al. [23] for a pair of mappings to G-metric spaces. Some comparative examples are constructed which illustrate the obtained results.

Keywords: Common fixed point, G-metric spaces, Weakly compatible, Expansive mapping.

1. Introduction

Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations and integral equations [24], game theory relevant to military, sports and medicine as well as economics [3]. A metrical common fixed point theorem is broadly comprised of conditions on commutativity, continuity, completeness and contraction besides suitable containment of range of one map into the range of the other. For proving new results, the researchers of this domain are required to improve one or more of these conditions. With a view to accommodate a wider class of mappings in the context of common fixed point theorems, Sessa [21] introduced the notion of weakly commuting mappings which was further generalized by Jungck [4] by defining compatible mappings. After this, there came a host of such definitions which are scattered throughout the recent literature whose survey and illustration (upto 2001) is available in Murthy [11]. A minimal condition merely requiring the commutativity at the set of coincidence points of the pair called weak compatibility was introduced by Jungck [6] in 1996. This new notion was extensively utilized to prove new results.

Mustafa and Sims [16] introduced the G-metric spaces as a generalization of the notion of metric spaces. Mustafa et al. ([12]-[15], [17]) obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of common fixed point in G-metric spaces.

In 1984, Wang et al. [23] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings in [18]. Rhoades [19] and Taniguchi [22] generalized the results of Wang [23] for pair of mappings. Later, Khan et al. [9] in 1986 generalized the result of [23] by making use of the functions. Kang [7] generalized these results of Khan et al. [9], Rhoades [19] and Taniguchi [22] for expansion mappings. In 2009, Ahmed [2] established a common fixed point theorem for expansive mappings by using the concept of compatibility of type (A) in 2-metric spaces. The theorem proved by Ahmed [2] was the generalization of the result of Kang et al. [8] for expansive mappings. Recently, Şahin and Telci [20] presented a common fixed point theorem for expansion type mappings in complete cone metric spaces which generalizes and extends the theorem of Wang et al. [23] for a pair of mappings to cone metric spaces.

The purpose of this paper is to generalize the results of Ahmed [2] to G-metric spaces by removing the condition of sequential continuity of the mappings. In order to prove the results, a more generalized concept of weak compatibility in G-metric spaces have been used instead of compatibility of type (A) used by Ahmed [2] in 2-metric spaces. Also, we extend the results of Şahin and Telci [20] to G-metric spaces thereby extending the theorem of Wang et al. [23] for a pair of mappings to G-metric spaces.

2. Preliminaries

Consistent with Mustafa and Sims [16], the following definitions and results will be needed in the sequel.

Definition 2.1: (G-Metric Space [16]).
Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties:

(1) $G(x, y, z) = 0$ *if* $x = y = z$,

(2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

(3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (*symmetry in all three variables*)

(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ (*rectangle inequality*).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2: ([16]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $\{x_n\} \to x$ in a G-metric space (X, G), then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

It has been shown in [16] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Proposition 2.1: ([16]). Let (X, G) be a G-metric space, then the following are equivalent:

(1) $\{x_n\}$ *is* $G-$*convergent to* x.

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0$ *as* $n \to \infty$.

Definition 2.3: ([16]). Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Proposition 2.2: Let (X, G) be a G-metric space. Then the following statements are equivalent:

(1) The sequence $\{x_n\}$ is G-Cauchy,

(2) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.

Definition 2.4: ([16]). Let (X, G), (X', G') be two G-metric spaces. Then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Definition 2.5: ([16]). A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.