Solving the Partial Differential Problems Using Maple

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ABSTRACT

This paper considers the partial differential problem of two types of multivariable functions and uses mathematical software Maple for verification. The infinite series forms of any order partial derivatives of these two types of multivariable functions can be obtained using binomial series and differentiation term by term theorem, which greatly reduce the difficulty of calculating their higher order partial derivative values. On the other hand, four examples are used to demonstrate the calculations.

Keywords: Partial derivatives, infinite series forms, binomial series, differentiation term by term theorem, Maple.

1. Introduction

In calculus and engineering mathematics, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. The Laplace equation, the wave equation, and other important physical equations involve the partial derivatives. The evaluation of the m -th order partial derivative value of a multivariable function at some point, generally, requires two procedures: the determination of the m -th order partial derivative of the function, and the substitution of the point into the m -th order partial derivative. These two procedures become increasingly complex calculations for increasing order of partial derivative, thus manual calculations become difficult. The present study considers the partial differential problem of the following two types of n -variables functions

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b \prod_{i=1}^{n} x_i^{\lambda_i} \right)^r \quad (1)$$

$$g(x_1, x_2, \cdots, x_n) = \exp\left(\sum_{i=1}^{n} \beta_i x_i \right) \cdot \left(a + b \exp\left(\sum_{i=1}^{n} \lambda_i x_i \right) \right)^r \quad (2)$$

where n is a positive integer, $a, b, r, \beta_i, \lambda_i$ are real numbers for all $i = 1,..,n$, $a, b \neq 0$, and a^r, b^r exist. We can obtain the infinite series forms of any order partial derivatives of these two types of multivariable functions using binomial series and differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), which greatly reduce the difficulty of calculate their higher order partial derivative values. The study of partial differential problems can refer to [1-24]. The methods adopted in [1-5] are different from the methods used in this paper, and [6-24] studied the evaluation of the partial derivatives of different types of multivariable functions using differentiation term by term theorem and complex power series method. [25] considered two differential equations whose independent variables involve the partial derivatives. [26] discussed the distance functions whose expressions contain the partial derivatives, and [27] found the solutions of some type of partial differential equation. In this article, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2. Main Results

Some notations used in this paper are introduced below.

2.1 Notations

2.1.1

$\prod_{i=1}^{n} c_i = c_1 \times c_2 \times \cdots \times c_n$, where n is a positive integer, c_i are real numbers for all $i = 1,..,n$.

2.1.2

Suppose that t is any real number, and m is any positive integer. Define

$(t)_m = t(t-1)\cdots(t-m+1)$, and $(t)_0 = 1$.

2.1.3

Suppose that n is a positive integer, j_i are non-negative integers for all $i = 1,..,n$. For the n -variables function $f(x_1, x_2, \cdots, x_n)$, its j_i times partial derivative with respect to x_i for all $i = 1,..,n$, forms a $j_1 + j_2 + \cdots + j_n$ -th order partial derivative, denoted as $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$

The followings are two important theorems used in this study.

2.2 Binomial series

$(1+u)^r = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} u^k$, where u, r are real numbers, and $|u| < 1$.

2.3 Differentiation term by term theorem ([28, p230]).

For all non-negative integers k, if the functions $g_k : (a,b) \to R$ satisfy the following three conditions: (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on the open interval (a,b) , and (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b) , then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) . Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

The following is the first major result in this study, we determine the infinite series forms of any order partial derivatives of the n -variables function (1).

2.4 Theorem 1

Suppose that n is a positive integer, $a, b, r, \lambda_i, \beta_i$ are real numbers for all $i = 1,..,n$, $a, b \neq 0$, and a^r, b^r exist. If the n -variables function

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b \prod_{i=1}^{n} x_i^{\lambda_i} \right)^r$$

satisfies that $x_i^{\beta_i}, x_i^{\lambda_i}, x_i^{\lambda_i r}$ exist, $x_i \neq 0$ for all $i = 1,..,n$, and $\prod_{i=1}^{n} x_i^{\lambda_i} \neq \pm \frac{a}{b}$.

Case A. If $\left| \prod_{i=1}^{n} x_i^{\lambda_i} \right| < \left| \frac{a}{b} \right|$, then the $j_1 + j_2 + \cdots + j_n$ -th order partial derivative of $f(x_1, x_2, \cdots, x_n)$

$$\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$$

$$= a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!} \left(\frac{b}{a} \right)^k \prod_{i=1}^{n} (\lambda_i k + \beta_i)_{j_i} \cdot \prod_{i=1}^{n} x_i^{\lambda_i k + \beta_i - j_i} \quad (3)$$

Case B. If $\left| \prod_{i=1}^{n} x_i^{\lambda_i} \right| > \left| \frac{a}{b} \right|$ $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$