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The Method of Images applied to the grounded sphere: The problem of the ground wire

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A B S T R A C T

The Method of Images poses an important difficulty when used to solve the problem of a charge in the presence of a grounded conducting sphere. This arises from the fact that the sum of the inducing charge and the image charge is different from zero. As a consequence, there is a monopole field far from the system, and any ground wire physically connected to the sphere will carry an electric current, changing the initial balance of charges until a new equilibrium is reached. The approach taken in this paper assumed an infinite straight wire connecting the sphere to ground. The charge distribution over the surface of the conductors was calculated, and the results analyzed. It was shown that the thinner the wire, the lower will be its total charge, and the closer will be the calculated charge density at the surface of the sphere to the conventional solution by the Method of Images.

1. Introduction

The Method of Images was first presented in 1849, in the early days of the construction of electromagnetic theory, by Sir William Thomson [1], later Lord Kelvin. It is still taught in the introductory discipline of electromagnetism, at the beginning of electrical engineering courses. It is based on the uniqueness of the solution of the Laplace Equation in a region limited by equipotential surfaces and is a powerful tool to solve a class of electrostatic problems, restricted to specific symmetries such as the infinite plane, the infinite cylinder, the sphere or, less commonly, the ellipsoid. For a more extensive discussion about the admissible regions for application of the Method of Images, see the work of J. B. Keller [2].

For a conducting sphere of radius a kept at zero potential, when an inducing charge Q is placed at a distance d from its center, the image charge must have an intensity of $(-a/d)Q$ and be placed at a distance (a^2/d) from the sphere center, on the straight line connecting the sphere center to the inducing charge. The sum of the potentials due to the inducing charge and to the image charge corresponds to an electric monopole $Q(1 - a/d)$, superposed onto an electric dipole moment with intensity $Qa/d(d - a^2/d)$. The sphere is located in the region where this sum vanishes. Fig. 1 shows a presentation of the method as usually found in textbooks.

It is usual to say that the zero potential condition in the conductor is obtained by connecting it to ground. However, the presence of the wire disturbs the solution of the problem.

Some amount of charge is needed to produce a zero potential on the wire surface, since it is a conductor at the same potential as the sphere. If the charge distribution over the wire surface is not taken into account, the electric field intensity in regions far from the sphere is proportional to a non zero monopole term, i.e.,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{(1-a/d)Q}{r^2}\frac{\mathbf{r}}{r}.$$

In that case, if a wire connects the sphere to ground, there will be a radial field along the wire, and an electric current will flow through it until the electrostatic balance is reached. The current will change the charge on the sphere surface, and its final value will be different from the value of the image charge indicated above.

Most textbooks disregard this effect. Authors say that the sphere is grounded to provide a source for the image charge, but most texts seem unconcerned about the consequences of physically adding a wire to the system [3–12]. J. D. Jackson [13] discusses the “idealizations that do not exist in the physical world” (Introduction, Section I.6, “Some Remarks on Idealizations in Electromagnetism”). He comments on the original discussion about electrical connections by thin wires by J. C. Maxwell [14], arguing that a very thin wire will need a correspondingly small amount of charge, which hardly disturbs the field at all, except in its immediate neighborhood,

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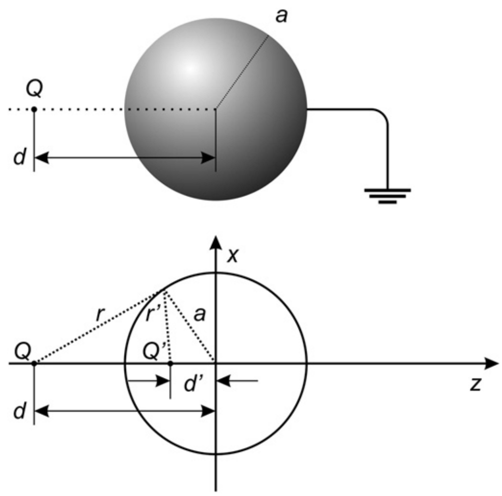

Fig. 1. Usual presentation of the Method of Images. On top, a conducting grounded sphere in the presence of the inducing charge Q. On the bottom, the corresponding schematic.

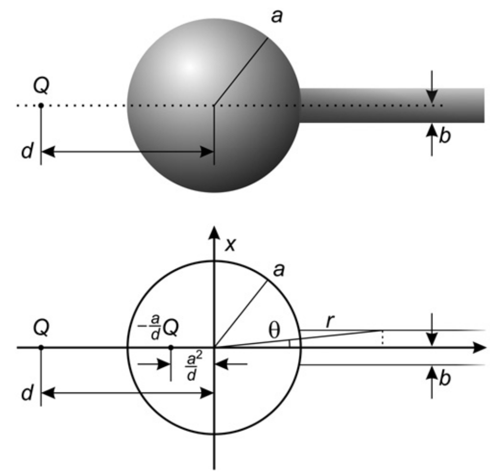

Fig. 2. Conducting sphere in the presence of an inducing charge, connected to earth by a straight wire. The wire is represented by a conductive cylinder of radius b in the positive side of the z-axis.

where the disturbance can be very large. That was the only reference we found discussing the presence of the ground wire.

For an engineering course, it is very important to emphasize the differences between a model and a real problem. The application of the Method of Images to the sphere is a good example to be explored by instructors in the discipline of electromagnetism to bring home this point.

This paper deals with the problem of a sphere connected to the ground by a wire in the presence of an inducing charge. The solution was obtained by two different methods, demonstrating that Maxwell's predictions were right.

It is interesting to see the facsimile edition of Maxwell's "Treatise on Electricity and Magnetism" [15]. It is a reproduction of the second edition (1881), and the owner of the original copy marked the text referring to the Method of Images applied to a conducting sphere, where it is written that the sphere *"is maintained at potential zero by connection with the earth"* (page 231). It seems that concerns about ground wires are nearly as old as the electromagnetic theory itself.

2. Theory

In the Method of Images, the sphere is simply assumed to be always at zero potential; textbooks mention no other conductors in the geometry, even as they say that the sphere is grounded. Now, consider instead that the sphere is connected to ground by an infinite straight wire. Referring to Fig. 2, a point charge Q is located at position $-d$ on the z-axis, and the conductive sphere of radius a is centered at the origin. The ground wire is assumed to be a cylinder of radius b, centered in the z-axis, and extending from the sphere to the infinite. The ground is able to supply any amount of charge without changing its own potential, which we take to be zero.

The system is axially symmetrical around the z-axis, suggesting the use of cylindrical or spherical coordinates. However, the conductive surface has no translational symmetry along the z-axis, and no spherical symmetry in the angular coordinate θ, making the problem substantially more difficult. The present work looks into the superficial charge density $\sigma(z)$, which, by symmetry, depends only on the z coordinate, or alternatively on the θ spherical coordinate.

Two methods were used to find the correct value of $\sigma(z)$ on conductor surfaces. The first one, based on the Method of Images, takes into consideration the image of the cylindrical wire on the sphere. The charge density on the wire surface, in turn, is determined by applying the boundary condition to its surface, i.e., zero potential. In the remainder of this paper, this method is referred to as *"Method of Green's Functions"*, to avoid confusion with the expression "Method of Images" as used in textbooks, since the same equations can be reached by the use of Green's functions for spherical surfaces.

The second method is based on the division of the conductor into an infinite number of rings. The idea is to find the charge of each ring as to satisfy the boundary conditions. This method, called in this paper "Method of Rings", is applicable to a class of geometries limited by the requirement that the locus of the surface be determined by a single variable, in our case either the cylindrical z or else the spherical θ coordinate.

In both methods, σ satisfies the zero potential condition ($\Phi = 0$) at the surface of the conductors, or $\Phi = \Phi_\sigma + \Phi_Q$, where Φ_σ is the potential due to the superficial charge distribution, and Φ_Q is the potential due to the inducing charge Q. Note that Φ_σ includes all image charge effects. Therefore, the condition to be satisfied is

$$\Phi_\sigma = -\Phi_Q(\text{at surface of conductors}), \tag{1}$$

where

$$\Phi_Q = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{r^2 + 2rd\cos\theta + d^2}}. \tag{2}$$

For a graphical interpretation of r and θ, see Fig. 2.

2.1. Solution by the Method of Green's Functions

As previously mentioned, the Method of Images applied to the case of a conducting sphere states that a zero potential on the surface of the sphere will occur if, for every charge Q_k in a position $\mathbf{r}_k$ outside the sphere of radius a, a corresponding image charge of intensity $-(a/r_k)Q_k$ is placed inside the sphere at position $(a^2/r_k)(\mathbf{r}_k/r_k)$. Thus, for each charge element $\sigma b d\phi dz$ of the wire, for a surface charge density σ, there must be a corresponding infinitesimal image charge $-(a/r)\sigma b d\phi dz$ in order to produce a null potential on the sphere surface. The potential due to the charge density on the wire $\Phi_{\text{wire}}(r,\theta)$, using spherical coordinates r,θ, is given by