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New solutions for charge distribution on conductor surface

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ABSTRACT

In the paper, a problem of electrostatics for charge distribution on a conductor surface is analytically solved for three new particular cases of conducting surfaces with complicated shape and specified value of electrostatic potential. The exact analytical expressions for surface charge density for the bodies are obtained. All the solutions are represented in a clear view of 3D graphs. It is shown that the proposed method of electrostatic problem for conductors allows to obtain infinitely many numerical solutions for the problem but only several special cases can be solved analytically.

1. Introduction

Let us consider a charged conducting body with an arbitrary shape in vacuum. One can obtain a uniquely determined expression for electrostatic potential ϕ governed with the body charge in a space point by solving Dirichlet problem for Laplace equation [1]:

$$\Delta\varphi = 0, \tag{1}$$

$$\varphi|_{\Sigma} = \varphi_1, \tag{2}$$

$$\varphi|_{\infty} = 0 \tag{3}$$

where φ_1 is a certain constant. According to Equation (3), the electric potential at infinity is chosen to be zero.

The problem has a set of analytical solutions and most of them are given in classical electrodynamics textbooks [2–5], e.g. a charged conducting ellipsoid and its particular cases. There are analytical solutions that represent a surface of two intersecting spheres [6]. A class of solutions derived by means of electrostatic image method and in terms of complex potential [7,8] exists. There are some elegant solutions for a uniformly charged elliptic ring [9], two conducting spheres [10], a uniformly charged square [11] and for a uniformly charged rectangular shape [12]. The mentioned analytical solutions are of great concern for electrostatics problem since they enable one to analyze efficiency of its different numerical solutions and serve as a beacon for qualitative understanding of electrostatic charge distribution on a surface of different conducting shapes.

A new class of non-trivial solutions for the electrostatics of conductors is investigated in the paper. It is derived from a well-known presentation of a solution for the Laplace equation as an expansion in terms of spherical harmonics [13]:

$$\varphi(r,\theta,\phi) = \sum_{n=0}^{+\infty}\sum_{k=-n}^{n}\frac{a_{nk}}{r^{n+1}}\cdot Y_n^k(\theta,\phi), \tag{4}$$

where $Y_n^k(\theta,\phi)$ are spherical harmonics.

In particular, Lord Rayleigh used such expansion of a potential in series of spherical harmonics to investigate an instability of a charged drop with an arbitrarily deformed surface [14].

Every term of the infinite series

$$\sum_{k=-n}^{n}\frac{a_{nk}}{r^{n+1}}\cdot Y_n^k(\theta,\phi)$$

is a particular solution of the Laplace equation [15].

And any finite sum of the series (5) is also a solution for the Laplace equation

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$$\varphi(r,\theta,\phi)=\sum_{n=0}^{N}\sum_{k=-n}^{n}\frac{a_{nk}}{r^{n+1}}\cdot Y_n^k(\theta,\phi) \tag{5}$$

One can use the fact in order to obtain new analytical solutions for the electrostatics problems of conductors.

2. The problem-solving method

In order to derive an analytical solution, let us choose such special shape of a charged conducting surface Σ that coincides with one of the equipotential surfaces governed with Equation (5) at a constant potential value φ_Σ.

A solution for the outer boundary Dirichlet problem (1)–(3) will be obtained when one fits coefficients a_{nk} such as the potential (5) is equal to the specified value φ_Σ in any point of the closed conducting surface Σ i.e.

$$\varphi_\Sigma=\sum_{n=0}^{N}\sum_{k=-n}^{n}\frac{a_{nk}}{r^{n+1}}\cdot Y_n^k(\theta,\phi) \tag{6}$$

Expression (6) is the $(N+1)$th degree polynomial with respect to reciprocal radius $1/r$. Thus, it is necessary to solve a polynomial equation with the coefficients as a linear combination of spherical functions $Y_n^k(\theta,\phi)$ to obtain the feasible shape of the closed equipotential surface. Any solution is a function of spherical coordinates θ, ϕ and governs an equipotential surface

$$r=r(\theta,\phi). \tag{7}$$

There are $N+1$ different solutions in a general way. But the real solution governing a closed surface for specified coefficients a_{nk} is single in accordance with a uniqueness of the solution for the electrostatics problem. Mathematical demonstration of the fact is a separate issue for the polynomials with the specified coefficients. It should be pointed out that an analytical solution for the Equation (6) exists for polynomials solvable by quadratures i.e. up to the 4th degree only when $N\le 3$. Otherwise the roots are transcendental and a numerical solution is available only.

Thus there are four sets of possible analytical solutions for Dirichlet problem (1)–(3) for closed shapes derived with Equation (6). A surface of the shapes (7) is specified with analytical expressions for solutions of respective polynomial equations.

When the potential depends on polar angle θ only and does not on azimuthal angle ϕ (in axially symmetric case) the derived surfaces are surfaces of revolution with respect to Z-axis. They are determined with the following equation [15].

$$\varphi_\Sigma(r,\theta)=\sum_{n=0}^{N}a_n\frac{P_n(\cos\theta)}{r^{n+1}} \tag{8}$$

Where $N\le 3$, $P_n(cos\theta)$ are the nth degree Legendre polynomials, $a_n=a_{n0}$ in series (5). In particular, it is a uniformly charged sphere when $N=0$. Let us thoroughly examine some particular cases with $N=1$ and $N=2$.

3. The surface equation and the charge distribution in case with $N=1$

Let us consider the case when $N=1$ in the series (8). Then there are only two terms in the expression for the electrostatic potential

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{a_0}{r}\pm\frac{a_1}{r^2}P_1(\cos\theta)\right)=\varphi_\Sigma,\ \phi\in[0;2\pi], \tag{9}$$

where ε_0 is the dielectric constant, a_1 and a_0 are arbitrary constants, φ_Σ is a constant potential on the shape surface. Parameter a_0 has the meaning of a total electric charge q distributed on the surface specified with Equation (9).

In this case, the conductor surface and the surface charge distribution are represented as surfaces of revolution about the Z-axis though non-symmetrical with respect to the X and Y axes as it is shown below.

In order to rewrite Equation (9) in a dimensionless form let us use the following notations

$$\varphi_0=\frac{a_0}{4\pi\varepsilon_0 r_0},\quad \xi=r/r_0,\quad \Psi=\frac{\varphi_\Sigma}{\varphi_0}=\frac{\varphi_\Sigma}{a_0/4\pi\varepsilon_0 r_0} \tag{10}$$

for the electrostatic potential of a conducting sphere with radius r_0 and total charge $q=a_0$, the radius-vector normalized with the sphere radius and the dimensionless potential respectively.

Let us divide the two sides of Equation (9) by potential φ_0.

Then surface Equation (9) takes the following dimensionless form:

$$\Psi\xi^2-\xi\mp k_1P_1(\cos\theta)=0, \tag{11}$$

where

$$k_1=a_1/(a_0r_0) \tag{12}$$

is a dimensionless coefficient.

The surface shapes obtained from Equation (11) when one choose signs «+» and «−» are congruous by reflection. Let us solve Equation (11) with sign «−».

Since $\xi(\theta)$ is an absolute value of a dimensionless radius-vector, only its nonnegative real values are physically meaningful. The Equation (11) has two roots but only one is physically correct. Thus, we obtain the following equation for the surface in terms of dimensionless spherical coordinate ξ (10) derived above:

$$\xi(\theta)=\frac{1+\sqrt{1+4\Psi k_1\cos\theta}}{2\Psi},\quad \phi\in[0;2\pi], \tag{13}$$

where Ψk_1 is to satisfy the following relation

$$\Psi k_1\le 1/4. \tag{14}$$

In a case when a value of Ψk_1 does not satisfy the relation (14), $\xi(\theta)$ is a complex value for some θ and the resulting surface has discontinuities.

Let us investigate the charge distribution on the surface (13).

It is well known that the electric field from a conductor close to its surface equals

$$E=\frac{\sigma}{\varepsilon_0}=|\vec{\nabla}\varphi|, \tag{15}$$

where σ is the surface charge density.

Let us use the following notation for the surface charge density on a spherical conductor with radius r_0 and total charge $q=a_0$ as in Equation (11)

$$\sigma_0=\frac{a_0}{4\pi r_0^2} \tag{16}$$

Then the surface charge density derived from Equation (15) can be rewritten in a dimensionless form

$$\tilde{\sigma}=\frac{\sigma}{\sigma_0}=\sqrt{\left(\frac{d\Psi}{d\xi}\right)^2+\frac{1}{\xi^2}\cdot\left(\frac{d\Psi}{d\theta}\right)^2} \tag{17}$$