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The Green's function of the Poisson equation on the non-concentric annular region

André Liemert*

Institut für Lasertechnologien in der Medizin und Meßtechnik an der Universität Ulm, Helmholtzstr. 12, D-89081 Ulm, Germany

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ABSTRACT

In this article the two-dimensional Poisson equation is considered in the region between two non-concentric circular cylinders. Upon introducing bipolar coordinates the corresponding Green's function is found in form of a simple and rapidly converging series which can be formally summarized as a closed-form. Based on this result we additionally provide the Green's function for the conducting cylinder which is oriented parallel to a ground plane as well as for the case of two conducting cylinders.

1. Introduction

The Poisson equation is involved in many areas of science such as electrostatics, optics, steady-state heat flow and computer graphics [1–3]. Analytical solutions of this equation can be derived for several geometries via the separation of variables [4], the application of certain integral transforms [5] or the method of images [6]. In the case of the two-dimensional (2D) Poisson equation the conformal mapping [1] provides a powerful tool to convert a relatively complicated boundary-value problem (BVP) into a simplified one. Well-known examples of this are the Möbius transformation and the Schwarz–Christoffel transformation [7]. Furthermore, Lekner recently derived analytical solutions of the 2D Laplace equation for the case of two conducting cylinders [10,11] and some time earlier on the region between the branches of a hyperbola [8].

In this article we consider the 2D Poisson equation in the region between two non-concentric circular cylinders and derive the corresponding Green's function in terms of rapidly converging elementary functions as well as in the formal closed-form representation. The obtained result can be readily extended to obtain additionally the Green's function for the conducting cylinder which is oriented parallel to a ground plane as well as for two conducting cylinders. In literature, a Green's function solution of the Poisson equation for the region between two non-concentric circular cylinders is available in form of an eigenfunction expansion [9]. However, the evaluation of this infinite series at points close to the singularity becomes a challenging task due to numerical shortcomings. The Green's function expressions derived in this article have, to our knowledge, not been reported in literature so far. It is shown that these simple expressions which can be easily implemented within a MATLAB script converge rapidly and reliably to the exact solution.

2. The bipolar coordinates

The analytic form of the Green's function on the non-concentric annular region can be derived in a straightforward manner upon the introduction of appropriate curvilinear coordinates. The BVP in question is a classical example for the application of the bipolar coordinates (ξ, η) which are related to the Cartesian coordinates (x, y) according to [11]

$$x = \frac{f\sinh\eta}{\cosh\eta - \cos\xi}, \quad y = \frac{f\sin\xi}{\cosh\eta - \cos\xi}, \tag{1}$$

where $\xi \in (-\pi, \pi]$ and $\eta \in (-\infty, \infty)$. Defining the complex variables $z = x + iy$ and $w = \xi + i\eta$ we alternatively can write the above relation in the compact form $z = if\cot(w/2)$ with $f > 0$ being a positive constant which will be specified below. Using bipolar coordinates it can verified via elimination of ξ in (1) that curves of constant η are mapped onto the non-concentric circles

* Tel.: +49 731 1429223; fax: +49 731 1429442.
E-mail address: andre.liemert@ilm.uni-ulm.de.

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$$(x-f\coth\eta)^2+y^2 = \frac{f^2}{\sinh^2\eta}, \quad (2)$$

with radii $f/|\sinh\eta|$ and centers at $(f\coth\eta, 0)$. Note that for $\eta<0$ the corresponding circles are located in the left half plane $x<0$. Fig. 1 indicates the relation between lines of constant η and the corresponding non-concentric circles. It can also be seen that the interior of the rectangular region is conformal mapped into the region between two non-concentric circles.

In order to convert a point from Cartesian into bipolar coordinates we have to consider the inverse relation to (1) which is given by the inverse cotangent function

$$w = \xi + \mathrm{i}\eta = 2\mathrm{arccot}\left(\frac{z}{\mathrm{i}f}\right). \quad (3)$$

Moreover, the Laplacian and the δ-function in bipolar coordinates have the form

$$\nabla^2\psi = \frac{(\cosh\eta-\cos\xi)^2}{f^2}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}\right) \quad (4)$$

and

$$\delta(x-x_0) = \frac{(\cosh\eta-\cos\xi)^2}{f^2}\delta(\xi-\xi_0)\delta(\eta-\eta_0). \quad (5)$$

The above relations are used in the next section in order to solve the Poisson equation in bipolar coordinates.

3. The Green's function on the region between two non-concentric circles

In this section the Poisson equation

$$\nabla^2 G(x,x_0) = -\delta(x-x_0) \quad (6)$$

is considered on the annular region between the inner and outer circle $(x-f\coth\eta_1)^2+y^2=a^2$ and $(x-f\coth\eta_2)^2+y^2=b^2$, respectively. Note that the geometry of the problem is schematically shown in Fig. 1. The Green's function in question is the solution of (6) under the condition that G vanishes on both circles. Upon introducing bipolar coordinates we obtain the Poisson equation

$$\frac{\partial^2 G}{\partial\xi^2}+\frac{\partial^2 G}{\partial\eta^2} = -\delta(\xi-\xi_0)\delta(\eta-\eta_0), \quad (7)$$

on the rectangular region $\Omega=\{w\in\mathbb{C}|-\pi<\xi\le\pi, \eta_2\le\eta\le\eta_1\}$, where $G=G(w,w_0)$ vanishes on $\eta=\eta_1$ and also on $\eta=\eta_2$. Both conditions must be satisfied for all $\xi\in(-\pi,\pi]$. Note that the intended solution is not the Green's function on a rectangle subject to the Dirichlet BC $G|_{\partial\Omega}=0$ because in the present case we have $G\ne0$ on $\xi=\pm\pi$. The values of G on the horizontal boundaries of the rectangle are not yet known and hence it must be determined as follows. For a given inner radius a, outer radius b and distance d between both centers we can set the following conditions

$$\frac{f}{\sinh\eta_1} = a, \quad \frac{f}{\sinh\eta_2} = b \quad (8)$$

and $f(\coth\eta_2-\coth\eta_1)=d$. Upon solving these equations we find the upper and lower boundary of the rectangle as well as the focus f as function of the geometrical parameters according to

$$\eta_1 = \mathrm{arcosh}\left(\frac{b^2-a^2-d^2}{2ad}\right), \quad (9)$$

$$\eta_2 = \mathrm{arcosh}\left(\frac{b^2-a^2+d^2}{2bd}\right), \quad (10)$$

and $f=\sqrt{(b^2-a^2-d^2)^2-(2ad)^2}/(2d)$. The Green's function on the rectangular region can be expanded in terms of trigonometric eigenfunctions

$$G(w,w_0) = \frac{2}{\eta_1-\eta_2}\sum_{l=1}^{\infty}G_l(\xi,\xi_0)\sin\lambda_l(\eta-\eta_2)\sin\lambda_l(\eta_0-\eta_2), \quad (11)$$

where $\lambda_l=l\pi/(\eta_1-\eta_2)$ are the eigenvalues and

$$\delta(\eta-\eta_0) = \frac{2}{\eta_1-\eta_2}\sum_{l=1}^{\infty}\sin\lambda_l(\eta-\eta_2)\sin\lambda_l(\eta_0-\eta_2) \quad (12)$$

is the corresponding completeness relation. Upon inserting (11) and (12) in (7) results in the following ordinary differential equations for $G_l=G_l(\xi,\xi_0)$

$$\frac{\mathrm{d}^2G_l}{\mathrm{d}\xi^2}-\lambda_l^2G_l = -\delta(\xi-\xi_0), \quad -\pi<\xi\le\pi. \quad (13)$$

One possible solution of (13) can be given in form of the slowly converging Fourier series

$$G_l(\xi,\xi_0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{e^{\mathrm{i}m(\xi-\xi_0)}}{m^2+\lambda_l^2}, \quad (14)$$

which however can be summarized in closed-form. To this end the Dirac delta function in (13) is replaced by the 2π-periodic Dirac comb so that Equation (13) can be considered in the infinite domain $\xi\in(-\infty,\infty)$ and alternatively written as

$$\frac{\mathrm{d}^2G_l}{\mathrm{d}\xi^2}-\lambda_l^2G_l = -\sum_{k=-\infty}^{\infty}\delta(\xi-\xi_0-2k\pi). \quad (15)$$

The impulse response of the one-dimensional diffusion equation

$$\left(\partial_x^2-\kappa^2\right)G(x,x_0) = -\delta(x-x_0), \quad -\infty<x<\infty, \quad (16)$$

is known to be $G(x,x_0)=e^{-\kappa|x-x_0|}/(2\kappa)$ and hence it can be used for solving (15) via superposition

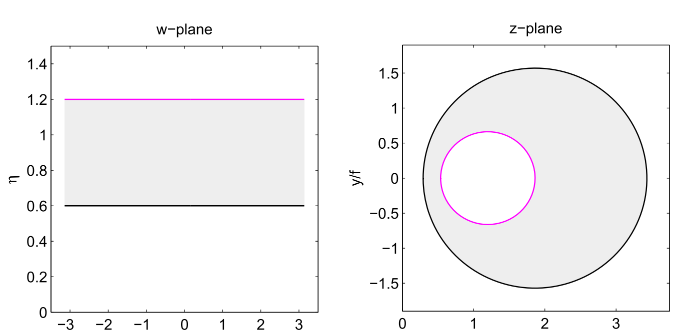

Fig. 1. Conformal mapping of a rectangular domain into the interior of a non-concentric annular region by means of the analytic function $z=\mathrm{i}f\cot(w/2)$.