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Distributions of potential and field on central axis line generated by elliptic ring uniformly charged

Ping Zhu[a,*], Yi Jie Zhu[b]

[a] Department of Physics, Simao Teacher's College, Puer 665000, China
[b] Faculty of Construction Management and Real Estate, Chongqing University, Chongqing 400045, China

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ABSTRACT

Using the theory of elliptic integrals and the mathematic trick, we obtain analytic distribution functions of the potential and the field on the central axis line generated by an elliptic ring uniformly charged and we make the discussion about it. The important properties of the potential and the field on the central axis line generated by an elliptic ring uniformly charged is shown.

1. Introduction

The field distribution of a uniformly charged circular ring is a important and interesting problem in electrostatics. In the research and the application of electromagnetism it is significant for us to investigate this problem [1–7], which has attracted considerable attention. However, because the problem involves elliptic integrals, it is quiet difficult for us to discuss and to solve the problem. Applying Fourier Bessel transform pair [8,9], Ball [10] presented the potential on the uniformly charged circular ring. Using computers and the superposition principle of the electric field, Zhang and Jiang [11] discussed the electric field distribution of annular linear electric charge. According to the superposition theorem of the potential and the field of a point charge, Cheng et al. [12] derived a series solutions of the potential and the field of a uniformly charged ring. In the rectangular coordinate system, Zhou and Chen [13] directly calculated the space distribution of the electrical field generated by a uniformly changed ring. In the spherical coordinate system and in the cylindrical coordinate system Zhu presented the spatial distribution of the field generate by a charged ring and the spatial distribution of the magnetic field generated by a current loop, respectively [14,15]. Making use of the analogous method, Trinh and Maruvada [16] investigated the electric field distribution of an arbitrary circular-arc shaped uniform current filament embedded in a conductive medium, the solution to which has been derived in terms of elliptic integrals and used in the numerical evaluation of the resistance of complex ground electrodes. Employing the theory of elliptical integrals and the analytic method, Zhu [17] obtained the analytic field distribution of a uniformly charged circular arc.

Nevertheless, for an elliptic ring, the polar radius of which ρ is not a constant and it does not possess symmetry of a circular ring, it is considerably difficult for us to solve the potential distribution and the field distribution of a uniformly charged elliptic ring and to discuss it. By now, the spatial distributions of the potential and the field with analytic methods are not drown. Just for this reason, it is specially interested to solve and to discuss the problem.

In this paper, by using the theory of elliptic integrals, properties of elliptic integrals and some special tricks, and transforming the elliptic equation, we can turn difficulty into simplicity, and gain the analytic potential function and the analytic field function on the central axis line generated by an elliptic ring uniformly charged, so that this problem is further solved. In Section 2, by the elliptic equation transformation and using properties of elliptic integral functions we present the potential distribution function of the uniformly charged elliptic ring on the central axis line. Similarly, in Section 3, we obtain the field distribution function of the uniformly charged elliptic ring on the central axis line. Finally, in Section 4 the

* Corresponding author. Tel.: +86 0879 2161220.
E-mail address: zhuupp@yahoo.com.cn (P. Zhu).

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distributions of the potential and the field on the central axis line attained are discussed and the important properties of the potential and the field distributions on the central axis line are drown. The summary and conclusion of the results include the paper.

2. Potential function on the central axis line of an elliptic ring uniformly charged

There is a uniformly charged rigid elliptic ring of which the semimajor axis is a, the semiminor axis is b, and the linear charge density is τ, and we will discuss its potential and field distributions in a cylindrical coordinate system, as depicted in Fig. 1.

In a cylindrical coordinate system the equation of the elliptic ring is given by

$$\rho = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}. \tag{1}$$

At a point on the elliptic ring, $A(\rho,\theta,0)$, we take a line element ds with electric quantity $dq = \tau ds$, which generates the potential at the point on the central axis line $P(0,0,z)$ is given by

$$d\varphi = \frac{dq}{4\pi\varepsilon_0\sqrt{z^2+\rho^2}} = \frac{\tau ds}{4\pi\varepsilon_0\sqrt{z^2+\rho^2}}. \tag{2}$$

In Eqs. (1) and (2), performing the variable transform $\psi = \theta + \pi/2$, we can obtain as follows

$$\rho = \frac{b}{\sqrt{1-e^2\sin^2\psi}}, \tag{3}$$

$$\begin{aligned} ds &= \sqrt{\rho^2 d\theta^2 + d\rho^2} = \sqrt{\rho^2 d\psi^2 + d\rho^2} \\ &= \sqrt{\rho^2 d\psi^2 + \frac{\rho^4 e^4 \sin^2\psi\cos^2\psi d\psi^2}{b^2\left(1-e^2\sin^2\psi\right)}} \\ &= \frac{\sqrt{1-(2e^2-e^4)\sin^2\psi}}{\left(1-e^2\sin^2\psi\right)}\rho d\psi \\ &= \frac{\sqrt{1-e'^2\sin^2\psi}}{\left(1-e^2\sin^2\psi\right)}\rho d\psi, \end{aligned} \tag{4}$$

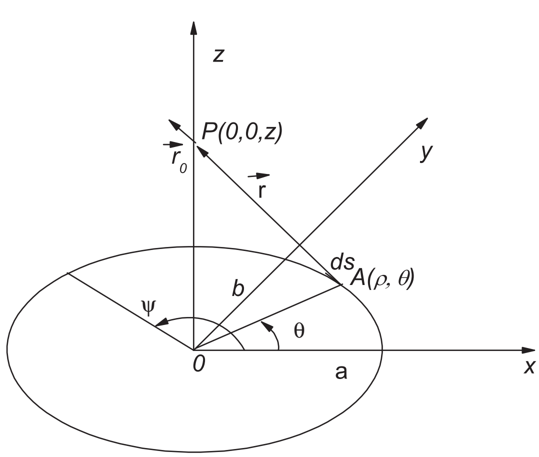

Fig. 1. A uniformly charged elliptic ring.

and

$$\begin{aligned} d\varphi &= \frac{\tau\sqrt{1-e'^2\sin^2\psi}}{4\pi\varepsilon_0\sqrt{1+\frac{z^2}{\rho^2}} \times \left(1-e^2\sin^2\psi\right)} d\psi \\ &= \frac{b\tau\sqrt{1-e'^2\sin^2\psi}}{4\pi\varepsilon_0\sqrt{b^2+z^2}\sqrt{1-\frac{z^2e^2}{z^2+b^2}\sin^2\psi} \times \left(1-e^2\sin^2\psi\right)} d\psi \\ &= \frac{b\tau\sqrt{1-e'^2\sin^2\psi}}{4\pi\varepsilon_0\sqrt{b^2+z^2}\sqrt{1-e''^2\sin^2\psi} \times \left(1-e^2\sin^2\psi\right)} d\psi, \end{aligned} \tag{5}$$

where

$$e^2 = \frac{a^2-b^2}{a^2} = 1-\left(\frac{b}{a}\right)^2 < 1, \tag{6}$$

$$e'^2 = 2e^2 - e^4 = 1-\left(\frac{b}{a}\right)^4 < 1, \tag{7}$$

and

$$e''^2 = \frac{z^2e^2}{z^2+b^2} < 1. \tag{8}$$

Then we have

$$\varphi = \frac{\tau b}{4\pi\varepsilon_0\sqrt{b^2+z^2}} \int_{\pi/2}^{5\pi/2} \frac{\sqrt{1-e'^2\sin^2\psi}}{\sqrt{1-e''^2\sin^2\psi}\left(1-e^2\sin^2\psi\right)} d\psi. \tag{9}$$

Obviously, the integral function of Eq. (9) $\sqrt{1-e'^2\sin^2\psi}/\sqrt{1-e''^2\sin^2\psi}(1-e^2\sin^2\psi)$ is a period function versus the variable ψ, whose period is 2π. Namely we have

$$\begin{aligned} &\int_{\pi/2}^{5\pi/2} \frac{\sqrt{1-e'^2\sin^2\psi}}{\sqrt{1-e''^2\sin^2\psi}\left(1-e^2\sin^2\psi\right)} d\psi \\ &= \int_{0}^{2\pi} \frac{\sqrt{1-e'^2\sin^2\psi}}{\sqrt{1-e''^2\sin^2\psi}\left(1-e^2\sin^2\psi\right)} d\psi. \end{aligned} \tag{10}$$

Thus Eq. (9) becomes

$$\varphi = \frac{\tau b}{\pi\varepsilon_0\sqrt{z^2+b^2}} \int_0^{\pi/2} \frac{\frac{1-e'^2\sin^2\psi}{1-e^2\sin^2\psi}}{\sqrt{\left(1-e'^2\sin^2\psi\right)\left(1-e''^2\sin^2\psi\right)}} d\psi. \tag{11}$$

Here $1 > e'^2 > e''^2 > 0$, we have [18]

$$\varphi = g\frac{\tau b}{\pi\varepsilon_0\sqrt{z^2+b^2}} \int_0^{u_1} R\left(\frac{sn^2u}{1-e''^2+e''^2sn^2u}\right) du, \tag{12}$$

where

$$k^2 = \frac{e'^2-e''^2}{1-e''^2},$$
$$sn^2u = \frac{(1-e''^2)\sin^2\psi}{1-e''^2\sin^2\psi},$$
$$g = \frac{1}{\sqrt{1-e''^2}},$$