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# Invariants in probabilistic reasoning

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ABSTRACT

Recent research has identified three invariants or identities that appear to hold in people's probabilistic reasoning: the QQ identity, the addition law identity, and the Bayes rule identity (Costello and Watts, 2014, 2016a, Fisher and Wolfe, 2014, Wang and Busemeyer, 2013, Wang et al., 2014). Each of these identities represent specific agreement with the requirements of normative probability theory; strikingly, these identities seem to hold in people's judgements despite the presence of strong and systematic biases against the requirements of normative probability theory in those very same judgements. These results suggest that the systematic biases seen in people's probabilistic reasoning follow mathematical rules: for these particular identities, these rules cause an overall cancellation of biases and so produce agreement with normative requirements. We assess two competing mathematical models of probabilistic reasoning (the 'probability theory plus noise' model and the 'quantum probability' model) in terms of their ability to account for this pattern of systematic biases and invariant identities.

## 1. Introduction

A fundamental goal of science is to find invariants: constant mathematical relationships that hold between different variables (Simon, 1990). Such invariants are perhaps the defining characteristic of quantitative 'hard' science: almost all important results in physics (Maxwell's equations of electromagnetism, Newton's law of gravitation, Einstein's field equations, the mass-energy equivalence, and so on) describe such invariant relationships. While such mathematical invariants or identities are rarer in the behavioural sciences, recent work has identified three identities that appear to hold in people's intuitive probabilistic reasoning: the 'QQ' ('Quantum Question') identity (Wang & Busemeyer, 2013; Wang, Solloway, Shiffrin, & Busemeyer, 2014), the addition law identity (Costello & Watts, 2014; Fisher & Wolfe, 2014), and the Bayes rule identity (Fisher & Wolfe, 2014; Costello & Watts, 2016a). Each identity describes a constant relationship that holds between different probabilistic judgements, and each represents specific agreement with the requirements of classical probability theory in those judgements. Strikingly, these identities or invariants hold in people's intuitive judgements of probability despite the presence of strong biases, or systematic deviations from the requirements of probability theory, in those very same judgements.

The fact that these mathematical identities appear to hold in people's probabilistic judgement (alongside patterns of systematic bias in those same judgements) has important implications for our understanding of how people reason about probability. It suggests that people judge probability in a way that follows some sort of formal, mathematical process that causes systematic biases in

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judgement whose values cancel each other out in particular invariant relationships. Indeed, these patterns of systematic bias alongside invariant identities are predicted by two competing formal models of probabilistic reasoning. One model, based on noisy frequentist probability, predicts the addition law and Bayes rule identities (Costello & Watts, 2014, 2016a) while the other, based on quantum probability, predicts the QQ identity (Wang & Busemeyer, 2013; Wang et al., 2014). In this paper we ask whether either of these models are able to account for all three invariant identities. In the first section we explain the QQ, addition law and Bayes rule identities. In the second section we present the quantum probability model and show that, while this model predicts the QQ identity, it is fundamentally unable to account for addition law and Bayes rule identity results. In the third section we present the 'probability theory plus noise' model and explain how it predicts the addition law and Bayes rule identity results. We also describe how priming effects in this model allow it to explain the QQ identity and to make some novel predictions connected to that identity (predictions that are supported by experimental results). In the fourth section we draw some general conclusions.

## 2. Identities in probabilistic reasoning

In presenting the addition law, Bayes rule and QQ identities we use the following notation, derived in part from quantum probability theory. We take $P(A)$ to represent the normatively correct probability of event $A$. We take **A** to represent a question about the occurrence or non-occurrence of $A$: in the language of quantum probability, **A** is an 'observable' that returns either $A$ or $\neg A$. The QQ identity is an invariant that relates answers to questions presented in different sequential orders. For tasks involving question ordering, we consider two possible orderings represented as **AB** (first a question about $A$, then a question about $B$) or **BA** (first a question about $B$, then a question about $A$). We take $P_{\mathbf{AB}}(A)$ to represent the subjective *estimated* probability of $A$ in the first ordering (the probability of getting a 'yes' answer to the question **A** when **A** is asked first and **B** is asked second). We take $P_{\mathbf{BA}}(A)$ to represent the subjective estimated probability of $A$ in the second ordering (the probability of getting a 'yes' answer to the question **A** when **B** is asked first and **A** second). We take $P_*(A)$ to represent the subjective estimated probability of $A$ when order is irrelevant (when **A** and **B** do not occur sequentially).[1] Since a subsequent presentation of question **B** cannot affect the results obtained from a prior presentation of question **A** (time travel is not allowed!), $P_*(A) = P_{\mathbf{AB}}(A)$ and $P_*(B) = P_{\mathbf{BA}}(B)$.

### 2.1. The QQ identity

Consider a situation where people are asked yes-no questions in two alternative orders **AB** or **BA**. This situation is commonly seen in polls; for example, in a Gallup poll conducted in September 1997, half of participants were asked the question "Do you think Al Gore is honest and trustworthy?" followed immediately by the question "Do you think Bill Clinton is honest and trustworthy?", while the other half of participants were asked the same questions in the reverse order (Moore, 2002). A noticable pattern of bias in such situations is that people's answers for a given question are often strongly influenced by the order of question presentation: the probability of a 'yes' answer to question **A** when that question comes first can be significantly different from the probability of a 'yes' answer when question **A** is preceded by question **B** ($P_{\mathbf{AB}}(A) \neq P_{\mathbf{BA}}(A)$). In the Clinton-Gore questions, for example, 76% of participants answered 'yes' to the Gore question when it was asked first (the **AB** order; $P_{\mathbf{AB}}(A) = 0.76$), while 66% answered yes when that question was asked second, after the Clinton question (the **BA** order; $P_{\mathbf{BA}}(A) = 0.66$): the prior presentation of the Clinton question produced a bias, reducing the likelihood of a 'yes' answer to the Gore question. These order effects occur both in polls on a range of different topics (Moore, 2002) and in similar experimental studies (Wang & Busemeyer, 2013; Wang et al., 2014).

Simultaneously, however, results (both from experimental studies and from poll data) show that the following identity tends to hold reliably in such sequential question answering:

$$P_{\mathbf{AB}}(A \wedge B) + P_{\mathbf{AB}}(\neg A \wedge \neg B) - P_{\mathbf{BA}}(A \wedge B) - P_{\mathbf{BA}}(\neg A \wedge \neg B) = 0$$

(this expression has a value of $-0.003$ in answers to the Clinton-Gore questions, for example). This represents unbiased agreement with the requirements of probability theory (in which, of course, the probability of a conjunction $A \wedge B$ does not depend on the order of the events within the conjunction). This identity appears to hold for all such consecutive questions, despite significant order effects for the same set of question answers. This identity holds for questions across a wide range of different topics in 72 different national representative surveys in the US, and in laboratory studies of the effects of order in question answering (Wang et al., 2014). This identity does not appear to hold when questions are not consecutive.

To connect the QQ identity more closely to order effects in sequential judgement, we rewrite it in the form

$$P_{\mathbf{AB}}(A \wedge B) - P_{\mathbf{BA}}(A \wedge B) = -[P_{\mathbf{AB}}(\neg A \wedge \neg B) - P_{\mathbf{BA}}(\neg A \wedge \neg B)] \quad (1)$$

This identity reveals an interesting pattern in the effects caused by question ordering: the contextual order effect observed when taking the difference between the probability of answering 'yes' to both questions in the order **AB** and the probability of answering 'yes' to both questions the order **BA** is equal to the negative of the contextual order effect observed when taking the difference between the probability of answering 'no' to both questions in the order **AB** and the probability of answering 'no' to both questions

[1] In previous work we've used $P_E(A)$ to represent this subjective estimated probability of $A$. We use the $P_*(A)$ notation here to stress the fact that this represents a subjective estimate in situations where ordering is irrelevant.