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Deformation monitoring and the maximum number of stable points method

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ABSTRACT

The question of determination of displacements in control networks with two or more measuring epochs is a well-known problem with broad applications to different fields of science and engineering. The standard procedure, which is computed by means of the pseudoinverse matrix, however, makes an implicit assumption that may be not convenient for the network at hand: it distributes the noticed displacement among the majority of the network points. The present paper develops what it has been named as the maximum number of stable points hypothesis and builds from the corresponding theoretical framework an applicable computation procedure. Application to a particular example will confirm its clear advantages versus the standard procedure for deformation determination in the cases where a single large deformation may be suspected.

1. Introduction

The theory and methods for deformation determination can be found in disparate areas of science and engineering, which include structural engineering, geodesy, surveying engineering, tectonics, geotechnical engineering and geomorphology, and may make use of observation techniques like Global Navigation Satellite Systems (GNSS) – e.g. Global Positioning System (GPS) – remote sensing, photogrammetry, Electronic Distance Measurement (EDM), angle measuring, levelling, etc. [1–10].

Control networks for deformation determination are generally classified into absolute and relative networks, depending whether they have points located outside the deformable area that can be considered stable (absolute networks) or all the network points may be affected by displacements so that only relative movements can be detected (relative networks) [7]. In the present paper, we will focus on the latter case, i.e. the case where all points are potential candidates for suffering a displacement between a pair of observing epochs.

In a deformation network redundant measurements are made among the control points for every epoch. Then the corresponding overdetermined systems of observing equations are formed and solved by least squares. Finally, the use of statistical tests over the least squares solution permits to conclude within the corresponding level of significance on the possible point displacements. This well-known theoretical framework will be developed in the next section. As it will be shown, no unique solution exists for the problem of determining relative displacements. In fact there are infinitely many solutions in terms of possible point displacements that are compatible with the observed values. In order to obtain a solution for the corresponding rank deficient systems the standard theory of deformation determination opts then for the pseudoinverse solution. As it will be argued, the pseudoinverse solution is a very sensible choice but it may not be the best option for all cases. The fact that the particular selected solution supposes to include an additional assumption directly affecting the results and what this assumption

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may be is a question often overlooked in deformation monitoring theory. The explanation of the assumption implicit in the adoption of the pseudoinverse solution (balanced distribution of displacements among all points) along with the proposal of a different assumption and correspondingly different solution arguably more sensible for other cases (stability of the majority of pillars and possible large displacements in very few of them) will be developed in the next section and later applied to the case of the open test field located in the *Universidad Politécnica de Valencia* campus.

2. Methods

2.1. Standard procedure for deformation determination

Let the system of observation equations for epoch 1 be written as

$$A_1x_1 = l_1 + r_1 \tag{1}$$

where A_1 denotes the coefficient matrix, x_1 the solution vector, l_1 the vector of independent terms, which includes the observed values, and r_1 the residual vector.

If observations are assumed to follow normal distributions with variance–covariance matrix Σ_1 then the residual vector also follows a normal distribution with zero mean and variance–covariance matrix Σ_1 and the most likely solution is obtained by the least squares condition arg $\min_{x_1} r_1^TP_1r_1$, where $P_1 = \Sigma_1^{-1}$, which leads to

$$(A_1^TP_1A_1)x_1 = A_1^TP_1l_1 \tag{2}$$

$$x_1 = (A_1^TP_1A_1)^{-1}A_1^TP_1l_1 \tag{3}$$

and analogously for epoch 2

$$x_2 = (A_2^TP_2A_2)^{-1}A_2^TP_2l_2 \tag{4}$$

The deformation vector is obtained then as

$$d = x_2 - x_1 \tag{5}$$

If the magnitudes observed in both epochs are the same (and ordered the same in both equation systems) and the same approximate coordinates are used in both epochs, Eq. (1) applied to both epochs can be written as

$$Ax_1 = l_1 + r_1 \tag{6}$$

$$Ax_2 = l_2 + r_2 \tag{7}$$

Subtracting Eq. (6) from Eq. (7) it can be written

$$Ad = l + r \tag{8}$$

where d is the deformation vector defined in Eq. (5), l is vector of differences between observed values in every epoch and r is the vector of differences between residuals.

Solution of the equation system in Eq. (8), named the observation differences model, is obviously equivalent to the separate solution of Eqs. (6) and (7) and subsequent determination of displacement by Eq. (5), and it will be preferred here for the sake of conciseness. The least squares solution verifies

$$(A^TPA)d = A^TPl \tag{9}$$

$$d = (A^TPA)^{-1}A^TPl \tag{10}$$

where $P = (\Sigma_1 + \Sigma_2)^{-1}$.

Now, as it was mentioned before, in the case of control networks where no point is free from potential displacements the corresponding system of equations is rank deficient. Therefore the matrix inverse in Eq. (10) – or inverses in Eqs. (3) and (4) if separate adjustments are preferred – does not exist. The solution is obtained then by means of generalized inverses, denoted by $(\cdot)^-$, in an equivalent manner

$$d = (A^TPA)^-A^TPl \tag{11}$$

where by definition, given a matrix $B \in \Re^{m\times n}$ a generalized inverse of B is a matrix $B^- \in \Re^{n\times m}$ that satisfies

$$BB^-B = B \tag{12}$$

The question now is that there are infinitely many generalized inverses, leading each of them to a different solution. There are then infinitely many deformation solutions that fulfill the least squares minimum condition, or in other words, there are infinitely many vectors d that satisfy Eq. (8) with the same residual vector r.

One particular generalized inverse is customarily selected: the pseudoinverse matrix. For any given matrix $B \in \Re^{m\times n}$ there always exists one and only one matrix – denoted by B^+ – that satisfies the four equations known as Moore–Penrose conditions [11]

$$BB^+B = B \tag{13}$$

$$B^+BB^+ = B^+ \tag{14}$$

$$(BB^+)^T = BB^+ \tag{15}$$

$$(B^+B)^T = B^+B \tag{16}$$

The pseudoinverse matrix is one of the generalized inverse matrices and it can be proved that has some *possibly* desirable properties that make it the habitual choice [12,13]: it is the generalized inverse of least determinant and least trace, and provides the solution of minimum L_2-norm among the infinitely many solutions of the system of equations.

The crucial point here is to acknowledge the assumption implicit in the customary selection of the pseudoinverse for the generalized inverse matrix in Eq. (11): the displacement vector d with least L_2-norm is being chosen as the best explicative solution to our rank deficient problem. In other words, in a control network where no point can be assured to remain stable we opt for the solution where Σd_i^2 is minimum, i.e. we prefer to explain the observed differences between epochs in the measured magnitudes as *small displacements of all points*, and this constitutes a working assumption incapable of being demonstrated or refuted (additional to those that led to select the least squares estimator as the best estimator).

One may argue that this hypothesis may appear to be reasonable for most cases. There are some occasions, however, where this assumption may go against the expected behavior, and therefore the customary use of the pseudoinverse solution must clearly be avoided. This is the case, for instance, of highly stable networks, such as the test field facility that we will see in Section 3, where clearly no displacements of the order of the observation