Measurement 50 (2014) 263–269

Contents lists available at ScienceDirect

Measurement

journal homepage: www.elsevier.com/locate/measurement

An effective quadrilateral membrane finite element based on the strain approach

Cherif Rebiai [b,a,*], Lamine Belounar [b,1]

[a] Mechanical Engineering Department, Superior National School of Technology, ENST SNVI National Road N°. 5, ZI, ROUIBA, Algiers, Algeria
[b] Numerical Modeling and Instrument in Soil-Structure Interaction Laboratory (MNIISS), University of Biskra, 07000 Biskra, Algeria

ARTICLE INFO

Article history:
Received 20 July 2013
Received in revised form 11 December 2013
Accepted 31 December 2013
Available online 12 January 2014

Keywords:
Strain approach
Elastoplastic analysis
Membrane finite element
Drilling rotation

ABSTRACT

Based on the strain approach, a new simple and efficient four-node quadrilateral membrane finite element with drilling rotation is developed. It can be used for the elastic and elastoplastic analysis. The displacements field of this element is based on the assumed functions for the various components of strain which satisfy the compatibility equation and it is developed in some way to improve the element performance in the distorted configurations. This finite element has the three degrees of freedom at each of the four nodes (the two translations and the in-plane rotation) and the displacement functions of the developed element satisfy the exact representation of the rigid body modes. Numerical results show that the proposed strain based element exhibits an excellent behavior for both regular and distorted mesh over a set of problems in both analyses.

1. Introduction

Since the appearance of finite elements method, many researchers have adopted the strain based approach for the development of new finite elements. The advantages of these elements have been illustrated on several articles [1,2] compared with displacement-based ones. The use of this approach was first applied by Ashwell and Sabir, and concerned only with curved elements [3,4].This approach was later extended to plane elasticity problems [5,6], to three-dimensional elasticity [7], to cylindrical shells [8–10], and to plate bending [11].

Also recently considerable attention has been given to the development of simple and efficient rectangular elements having the in-plane rotation as nodal degrees of freedom [12]. The main motivation is the improvement of the accuracy and to provide an ideal membrane element to form shell element. Indeed in the past several models such as rectangular and triangular plane elasticity elements were developed, among them the elements of Sabir [13] which each of them have three degrees of freedom (DOFs) at each corner node. However these developed strain based elements with drilling rotation are efficient only for the regular meshes.

In this context the proposed element in this paper is a new quadrilateral membrane finite element with drilling rotation based on the strain approach named SBQE (**S**train **B**ased **Q**uadrilateral **E**lement) able to improve the accuracy and the computation time in the case of regular and distorted mesh. Both linear and materially nonlinear analyses are considered. For the purposes of demonstration some selected numerical problems are solved using this developed element.

2. Formulation of the SBQE element

Consider a quadrilateral element SBQE with three degrees of freedom (U_i, V_i, and in plane rotation θ_i) at each of the four nodes which is depicted in Fig. 1.

* Corresponding author at: Mechanical Engineering Department, Superior National School of Technology, ENST SNVI National Road N°. 5, ZI, ROUIBA, Algeria. Tel./fax: +213 0776 01 18 50.
E-mail addresses: c_rebiai@yahoo.fr (C. Rebiai), belounarl@yahoo.com (L. Belounar).
[1] Tel./fax: +213 0666 25 61 45.

0263-2241/$ - see front matter © 2014 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.measurement.2013.12.043

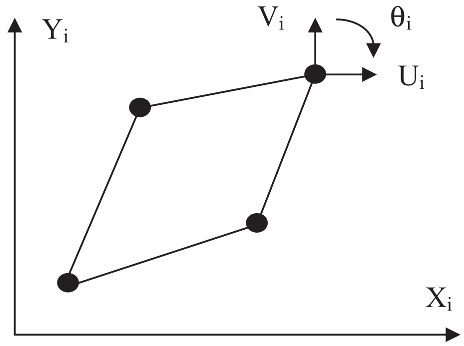

Fig. 1. SBQE element and coordinate system.

For general plane elasticity problems, the three components of strain in terms of the displacements are given by

$$\varepsilon_x = \frac{\partial U}{\partial x}, \quad \varepsilon_y = \frac{\partial V}{\partial y}, \quad \gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \tag{1}$$

The components of the strain given in Eq. (1) must satisfy an additional equation called the compatibility equation. This equation can be formed by the eliminating U, V from Eq. (1), hence:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \tag{2}$$

If these strains given by Eq. (1) are equal to zero, the integration of these equations allows obtaining the following expressions:

$$U = a_1 - a_3 y, \quad V = a_2 + a_3 x, \quad \theta = a_3 \tag{3}$$

The terms in Eq. (3) are those representing the rigid body modes. The present element possesses four nodes and three DOFs (U, V, θ) per node. Thus the displacement field must contain twelve independent constants. Three of them (a_1, a_2, a_3) are already used for the representation of the rigid body components, thus it remains nine constants (a_4, $a_5 \ldots a_{12}$) for expressing the displacement due to straining of the element. These are apportioned among the strains as:

$$\begin{aligned} \varepsilon_x &= a_4 + a_6 y + a_7 x + a_{10} y^2 + 2a_{11} x y^3 \\ \varepsilon_y &= a_7 + a_8 x + a_9 y - a_{10} x^2 - 2a_{11} y x^3 \\ \gamma_{xy} &= 2a_5(y+1) + 2a_6 x + 2a_7 x + 2a_8 y + a_9 y + 2a_{12} x \end{aligned} \tag{4}$$

The strains given by Eq. (4) satisfy the compatibility equation given by Eq. (2). Expressions (4) are equated to the equations in terms of U, V from Eq. (1) and the resulting equations are integrated, to give

$$\begin{aligned} U &= a_4 x + a_5(y + y^2) + a_6 xy + 0.5a_7 x^2 + 0.5a_8 y^2 \\ &\quad + 0.5a_9 y^2 + a_{10} x y^2 + a_{11} x^2 y^3 \\ V &= a_5 x + 0.5a_6 x^2 + a_7(x^2 + y) + a_8 xy + 0.5a_9 y^2 - a_{10} x^2 y \\ &\quad - a_{11} x^3 y^2 + a_{12} x^2 \\ \theta &= -2a_5 y + a_7 x - a_9 y - 2a_{10} xy - 3a_{11} x^2 y^2 + a_{12} x \end{aligned} \tag{5}$$

The final displacement functions are obtained by adding Eqs. (3) and (5) to obtain the following:

$$\begin{aligned} U &= a_1 - a_3 y + a_4 x + a_5(y + y^2) + a_6 xy + 0.5a_7 x^2 + 0.5a_8 y^2 \\ &\quad + 0.5a_9 y^2 + a_{10} x y^2 + a_{11} x^2 y^3 \\ V &= a_2 + a_3 x + a_5 x + 0.5a_6 x^2 + a_7(x^2 + y) + a_8 xy + 0.5a_9 y^2 \\ &\quad - a_{10} x^2 y - a_{11} x^3 y^2 + a_{12} x^2 \\ \theta &= a_3 - 2a_5 y + a_7 x - a_9 y - 2a_{10} xy - 3a_{11} x^2 y^2 + a_{12} x \end{aligned} \tag{6}$$

The displacement functions of the developed element SBQE given by Eq. (6) can be written in matrix form as:

$$\{U\} = [C]\{A\} \tag{7}$$

where the U is the nodal displacement vector, A is the constant parameters vector $\{a_i\} = 1 \ldots 12$ and the 12×12 transformation matrix $[C]$ is given in appendix.

The stiffness matrix $[K^e]$ can be calculated from the well known expression:

$$[K^e] = [C]^{-T}\left(\int\int [Q]^T[D][Q]dxdy\right)[C]^{-1} = [C]^{-T}[K_0][C]^{-1} \tag{8}$$

The determinant of the Jacobean matrix must also be evaluated because it is used in the transformed integrals as follow:

$$\int\int dxdy = \int_{-1}^{+1}\int_{-1}^{+1} \det|J| d\xi d\eta \tag{9}$$

Thus the matrix $[K_0]$ is numerically evaluated, and since the matrix $[C]$ of the developed element is not singular, its inverse can be also numerically evaluated and the element stiffness matrix $[K^e]$ can be obtained by:

$$[K^e] = [C]^{-T}\left(\int_{-1}^{1}\int_{-1}^{1} [Q]^T[D][Q]\det|J| d\xi d\eta\right)[C]^{-1} \tag{10}$$

where the strain matrix $[Q]$ and the elasticity matrix $[D]$ are given in appendix.

3. Linear numerical results from test examples

Before proceeding to the benchmark problems which are mainly extracted from literature when discussing the element SBQE with drilling DOFs, a brief notes on the elements to be compared are given:

- Q8: the eight nodes quadrilateral element with sixteen degrees of freedom (DOFs).
- Q6: the six node quadrilateral element with twelve DOFs.
- SBRIEIR and SBTIEIR: the four and three node strain based rectangular and triangular in-plane elements with in-plane rotation with twelve DOFs [13].
- SBT2V: The Improved three node strain based triangular in-plane element with drilling rotation with nine DOFs [14].
- HQ4-9β: Isostatic quadrilateral membrane finite element with drilling rotation [15].
- P5Sβ: Pian's hybrid element with four nodes [16].
- FRQ: Quadrilateral element based on fiber rotation [22].
- Quadrilateral element with drilling ITW DOFS [18].
- Quadrilateral element with drilling rotation Pimp [19].
- Q4: quadrilateral element with four nodes.