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Bayesian recursive estimation of linear dynamic system states from measurement information

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ABSTRACT

The evaluation of uncertainty in dynamic measurements has recently become a demanding issue. A Bayesian approach is employed here to derive the equations required to recursively generate the solution to the problem of estimating (and predicting) the states of linear dynamic systems. It is shown that this approach allows a derivation of Kalman's filtering algorithm which is more easily accessible to those involved with dynamic measurements. The complete time-varying Kalman filter is particularly useful when the linear dynamic system and/or signal statistics are time varying and also when optimum estimates are required from the very beginning.

1. Introduction

In most measurements, it is generally assumed that the value of the measurand is constant and unique. The *Guide to the Expression of Uncertainty in Measurement* (GUM) [1] and its supplement 1 (GUM S1) [2] have been conceived on this assumption.

However, in many measurements that are important for scientific and industrial applications the value of the measurand varies over time. The evaluation of uncertainty in dynamic measurements has therefore received an increasing attention in recent years [3–7]. The methods proposed are in general applicable to dynamic measurements made with linear time-invariant systems. The goal has been to suggest practical dynamic analysis aligned with the general principles of the widely accepted GUM. A different viewpoint which allows nonstationarity at the outset is adopted here. This is particularly useful when the dynamic system and/or signal statistics are time varying and also when optimum estimates are required from the very beginning.

Bayesian methods provide a general framework for the assessment of uncertainty in traditional 'stationary' metrology [8,9]. Mathematically, it is formulated in this way: one requires the posterior distribution for the measurand which tells one as much as it is possible to know (and no more) about the measurand from whatever information is available such as direct measurements, established relations with other quantities, expert judgment, physical plausibility, manufacturer's specifications, and calibration certificates. The estimate of the measurand and the uncertainty associated with the estimate are then obtained from the moments of that distribution.

There is no reason for the quantity 'time' not to be included in the above framework. In many estimation problems, especially those involving dynamic systems modeled in the state-space representation, observations are made sequentially in time and up-to-date state estimates are required. Bayesian methods also provide a rigorous general framework for dynamic state estimation problems. In this case, the Bayesian approach is to construct the posterior distribution for the state based on all the available information.

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The recursive solution to the discrete-time linear estimation problem was first published by Kalman [10]. The estimation algorithm is the so-called *Kalman filter*. Such filters allow nonstationarity, and their recursive form is easy to implement. The Kalman filtering algorithm made it possible to navigate precisely over long distances and time spans; it is used extensively in all navigation systems for deep-space exploration.

Historically, this signal estimation problem was viewed as *filtering* narrowband signals from wideband noise; hence the name "filtering" for signal estimation. The objective of the estimation was to minimize the mean square error (MSE) between the random signal and its estimator when observations of a related random process are available. Kalman used orthogonality to derive his filtering algorithm. His exposition in [11] summarized the contribution of his original paper, although many details are different. Kalman's equations have often been derived using innovations. The concept of innovations, or the unpredictable part, of observations was introduced by Kailath [12]. The equations have also been derived in other classical ways [13].

Kalman's equations were derived using a Bayesian approach for the first time in [14] and a similar approach was used in [15]. Methods that employ Bayes' theorem were also used to derive the equations [16,17]. A Bayesian derivation which it is believed is more easily accessible to those involved with dynamic measurements is presented here. It will be shown that for the linear-Gaussian estimation problem, the posterior distribution for the state remains Gaussian at every iteration of the filter, and the Kalman filter relations propagate and update the moments of the distribution. The estimate of the state and the uncertainty associated with the estimate then become available at each iteration of the filter.

This article is restricted to linear dynamic systems. For nonlinear or non-Gaussian problems there is no general analytic (closed-form) expression for the required posterior distribution and one needs to resort to computationally intensive methods. The latter are topics for future research.

The article is organized as follows. Linear dynamic models are introduced in Section 2. The equations required to recursively generate the solution to the estimation problem are derived in Section 3. The prediction problem is discussed in Section 4. The analysis is extended to the more general vector case in the next sections. The reader mainly interested in that case may start directly in Section 5. A summary is provided in Section 8.

2. Dynamic models – scalar case

Upper-case italicized Latin letters will denote scalar quantities. The possible values that they are deemed to assume will be denoted by lower-case italicized Greek letters. Lower-case italicized Latin letters are reserved for constants. Scalar variances will be generically referred to by lower-case italicized Greek letters.

Linear dynamic models are state-space models whose state unpredictable variations with time are described probabilistically. They are characterized by a pair of equations, named the observation and system equations [18,9], that is, respectively,

$$X_n = b_n Z_n + \varepsilon_n, \quad \varepsilon_n \sim N(0, \sigma_n^2) \tag{1}$$

$$Z_n = a_{n,n-1} Z_{n-1} + w_n, \quad w_n \sim N(0, \omega_n^2) \tag{2}$$

where n is the time index ($n = 1, 2, \ldots$), X_n is the observation sequence, b_n is a known series of constants describing the linear relationship between the state and the observation, Z_n is a Gauss-Markov sequence, or a first-order autoregressive sequence of unknown process states, w_n is the so-called system noise driving function, i.e., a noise sequence normally distributed with zero mean and known variance ω_n^2, $a_{n,n-1}$ is a known series of constants, which is a description of the known first-order difference equation of the system, ε_n is a noise sequence normally distributed with zero mean and known variance σ_n^2. It is assumed that both w_n and ε_n are identically and independently distributed (white noise) and mutually uncorrelated.

Here all participating quantities are regarded as random variables when there is uncertainty about their values – either because knowledge about them is imperfect or incomplete, or because they are subject to the unpredictable variations of experimentation. This position does not imply that the value of any of the participating quantities is 'variable' in the common sense of this word. The defining trait of the Bayesian approach is to treat all participating quantities as random variables whose probability density functions (pdf) encode and convey states of (incomplete) knowledge about them [19–22].

3. Estimation – scalar case

Let $d_n = \{d_{n-1}, X_n = x_n\}$, with d_0 describing the initial available information, including the values of $a_{n,n-1}$, b_n, ω_n^2 and σ_n^2, $\forall n$, which are supposed to be known. All the information d_{n-1} about the unknown state is encoded by the posterior pdf at $n-1$ and used to derive the new posterior once the data sample $X_n = x_n$ is received at n. It is shown in the sequel how to evolve from the posterior pdf at $n-1$ to the posterior at n.

Denote the posterior pdf at $n-1$ by

$$p_{Z_{n-1}}(\zeta_{n-1}|d_{n-1}) \propto \exp\left\{-\frac{1}{2v_{n-1}^2}(\zeta_{n-1} - \hat{\zeta}_{n-1})^2\right\} \tag{3}$$

It is assumed that initially at time $n = 1$, information concerning the state Z_0 was described in the form of a Gaussian probability distribution with known mean $\hat{\zeta}_0$ and variance v_0^2. The prior pdf at n will be

$$p_{Z_n}(\zeta_n|d_{n-1}) \propto \exp\left\{-\frac{1}{2\rho_n^2}(\zeta_n - \tilde{\zeta}_n)^2\right\} \tag{4}$$

with

$$\tilde{\zeta}_n = a_{n,n-1}\hat{\zeta}_{n-1} \tag{5}$$

$$\rho_n^2 = a_{n,n-1}^2 v_{n-1}^2 + \omega_n^2 \tag{6}$$

This follows immediately from (2) and the properties of the Gaussian distribution.

A sampling distribution with unknown location and scale parameters is assigned that describes the prior