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Algebraic approach to extensive measurement based on direct comparison

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ABSTRACT

The aim of this paper is to give an algebraic description of an extensive measurement (with error), which consists of a series of direct comparisons of all copies of the measured object with all copies of the reference. The start point is the axiomatisation of measurement operations in contrast with the axiomatisation of properties of the order relation. We propose "principles of measurement consistency" which describe an operation of ordering of comparisons results. This principle is more general than the assumption of transitivity or homothecity of precedence (or preference) relation. In the presented model, the final result of a measurement is described by four numbers which represent both the measurement value and the uncertainty with two components: systematic and non-systematic. The regular inexact measurement and the biased measurement are also considered as particular cases.

1. Introduction

A measurement is an empirical procedure which maps empirical objects to mathematical objects (number or n-tuple of numbers). Measurement of physical quantities is based essentially on comparisons, because every physical quantity is expressed with respect to a reference (the standard). The aim of this paper is to develop a mathematical description of inexact measurement based on comparison with a non-ideal reference. We use the term "inexact" for models of measurement which take into account measurement errors. We propose an algebraic description of measurement based on comparison without use of the probability theory. The purpose of such an approach is to axiomatise the measurement procedure. The measurement procedure consists of a few well-known components, but was never expressed as a set of axioms which allow to obtain numbers (four rational numbers in our model) describing a measured quantity when non-systematic errors are meaningful.

In this paper, a model of measurement (measurement by comparison) is constructed under the following assumptions:

1. The value of the measured quantity (mesurand) is determined with respect to a reference (measurement is relative).
2. Result of the measurement is obtained by comparisons of n copies of an object under measurement x (denoted by $n.x$) with m copies of the reference z (any chosen object). Each fraction $q = m/n$ (called comparison fraction) is characterised by a relation between $n.x$ and $m.z$.
3. As a result of a single comparison, the following types of relations between two objects x and y can be determined:
   - x are y comparable i.e. one of them is greater than the other $x \prec y$ or $y \prec x$
   - x are y incomparable i.e. we cannot say whether $x \prec y$ or $y \prec x$. Such a situation is denoted by $x \sim y$

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4. If an assumption called the "principle of measurement consistency" is satisfied, then we have five types of relations between $n.x$ and $m.z$: two comparable situations, one incomparable and two conditionally comparable (comparable for $kn.x$ and $km.z$, where k is a natural number). Therefore all the comparison fractions form five subsets.
5. These five classes of fractions are characterised by four numbers (four boundaries). Moreover, four natural classes of such fractions can be set (in some cases only two) what leads to four numbers representing the value of the measured inexact quantity. The algorithm for determining the measurement result described in this paper will be called "inexact measurement".
6. All elements which constitute a measurement are non-ideal (inexact). This elements are: objects under measurement, references, the measurement operations (comparisons and copying). These inexactness can be described by various models (probability theory, fuzzy sets), but in the algebraic approach all the information of inexactness are contained in the properties of the $\prec$ relation.

The relation $\prec$ describes the operation of comparison in a mathematical way. This relation is typically expressed as "less", but we will call $\prec$ a "precedence relation" due to the fact that in our approach we assume that this relation is asymmetric only. The principles of measurement consistency MC1 and MC2 play important role in our theory and give more general framework then the assumptions such as transitivity or homothecity.

A relation $x \sim y$ describes a situation when neither $x \prec y$ nor $y \prec x$ and is called incomparability. Incomparability is a situation when it is impossible to determine by one comparison whether x is less than y or y less than x. We will assume that $\sim$ is not transitive and hence is not an equivalence relation. Summing up, $\sim$ is a relation defined by means of $\prec$ as a set of pairs of objects which are incomparable (a comparator is unable to determine precedence).

Incomparability $\sim$ is sometimes described as a "balance" i.e. a comparator equilibrium. But "balance" gives us information about a measured value only if a measurement error is equal to zero. The relation $\sim$ can be a basis for a description of the ideal measurement model without measurement errors if the incomparability is transitive $\sim$.

We will begin our discussion (in Section 2) with a simple example of balance and ruler measurements when a measurement can be carried out as a comparison of one copy of the object under measurement with n copies of the given reference z. Farther we will discuss the situation when the object under measurement can be copied and when a relation $\prec$ is non-homothetic (which describes the effect of statistical decreasing of errors due to repeating of measurements).

The paper is organised as follows. In Section 2 we describe non-formally our main idea of determining the result of measurement from series of comparisons. Section 3 contains the main result of our paper, therefore in this chapter we repeat some main definitions. In this section we give the construction and properties of the final result of extensive measurement based on direct comparison, and we show that to perform an extensive measurement it is sufficient to establish that the measurement objects are $\mathbb{N}$-set-structure with asymmetric relation and with the principle of measurement consistency. In Sections 4 and 5 we define the regular inexact measurement and consider the biased measurement as particular cases of measurement based on the direct comparison. Some conclusions and discussions we give in Section 6.

2. How comparison gives measurement results

In this section we star description of our idea from the discussion of measurement by comparison using a simple examples, then we describe the principles of measurement consistency and relation to measurement uncertainty.

2.1. Measurement by comparison of one copy of an object – measurement with a metric ruler and a balance

Every act of a measurement (even a "simple" one) requires many comparisons. For example let us consider the measurements of weight by means of balance and lengths by use of a metric ruler.

In case of exact measurement a typical assumption is that a balance is a result of a comparison (balance, as well as any comparator, is built in such a way that it is possible to obtain a compensation). Hence a compensation measurement is based on determining such natural number n that $x \sim n.z$. This condition means that n copies of the reference z is balanced by the object x and hence we say that n is the value $f(x)$ of the measured quantity of x with respect to z. Here it seems that a measurement is just a determination of the state of balance. What more in case of the metric ruler it seems that measurement is done simply by a single comparison.

There are some reasons for which it is in general impossible to determine a result of measurement by only one comparison i.e. by a uniquely determined condition of balance (compensation) [11]:

1. In order to determine the balance of comparator a series of comparisons should made.
2. In order to determine boundaries of incomparabilities (systematic error and threshold) we have to determine the area of comparability (limits of precedence).
3. Incomparability is not transitive (for example [12]).[1]

In case of inexact measurements (for example when we measure properties of objects described as a random or fuzzy variables), the state of balance is not well-defined. We can determine only the precedence relation $\prec$, in other words we can see when a scale pan is predominating, or when the length is greater.

When mass or length of an object is measured, a series of comparisons is carried out. This leads to the

[1] An easy example of non-transitivity of incomparability is connected with a resolution threshold. For example it is possible that $a \sim b$ and $b \sim c$ but $a \prec c$ because a difference between a and c is greater then a resolution threshold.