Optics and Lasers in Engineering 77 (2016) 100–111

Contents lists available at ScienceDirect

Optics and Lasers in Engineering

journal homepage: www.elsevier.com/locate/optlaseng

Mathematical models utilized in the retrieval of displacement information encoded in fringe patterns

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ARTICLE INFO

Article history:
Received 2 May 2015
Received in revised form
12 July 2015
Accepted 28 July 2015
Available online 24 August 2015

Keywords:
Fringe pattern analysis
Basic mathematical models for detection and recovery of displacements
Phase and amplitude modulation
Hilbert transform
Multiphase methods and in-quadrature signals analysis

ABSTRACT

All the techniques that measure displacements, whether in the range of visible optics or any other form of field methods, require the presence of a carrier signal. A carrier signal is a wave form modulated (modified) by an input, deformation of the medium. A carrier is tagged to the medium under analysis and deforms with the medium. The wave form must be known both in the unmodulated and the modulated conditions. There are two basic mathematical models that can be utilized to decode the information contained in the carrier, phase modulation or frequency modulation, both are closely connected. Basic problems connected to the detection and recovery of displacement information that are common to all optical techniques will be analyzed in this paper, focusing on the general theory common to all the methods independently of the type of signal utilized. The aspects discussed are those that have practical impact in the process of data gathering and data processing.

1. Introduction to signal analysis of patterns that contain displacement information

Fundamental mathematical problems in the analysis of 2-D and 3-D images containing displacement information are: 1) data retrieving; 2) data analysis; 3) data processing. One must have a robust method to retrieve signal information contained in an image against stochastic and deterministic local deviations of the recorded information. As stated in [1], the phase of a signal is a robust representation of the signal in the presence of noise. In [1], it is shown that the information contained in a 2-D signal can be recovered to a great extent only on the basis of phase information by setting the amplitude to unit value. On the contrary, if one makes the phase zero and intensity is recovered, the signal cannot be reconstructed. This is a very important aspect of fringe pattern analysis, where following the classical approach in optics one assumes that displacement information is contained in the phase of recorded signals that take the form of a phasor. In the text that follows a one-dimensional approach is utilized, the reason for this approach will be given in Section 10 of the paper. For methods that measure displacements, the general equation of a fringe system for a 1-D signal is of the form [2–4]

$$I(x) = I_0 + I_1 \cos \Psi(x) \tag{1}$$

where I_0 is a background term that ideally should be a constant value for all x's. In actual applications it is assumed to be a slow changing term as a function of x. This means that in the FT power spectrum of the signal, I_0 must be a spike at the origin of coordinates of the frequency space. The term containing the displacement information is the second term of Eq. (1), a phase modulated (frequency modulated) sinusoidal signal restricted to the first harmonic. In the ideal model, I_1 (the amplitude of the first order harmonic of the signal) is a constant. However, in actual signals, it is also a function of x: the phase term $\Psi(x)$ that can be of the form

$$\Psi(x) = \left[2\pi f_c x + \phi(x)\right] \tag{2}$$

where f_c is the frequency of the carrier that is phase modulated by the term $\phi(x)$ containing the information on the optical path change caused by the displacement field. Due to the presence of $\phi(x)$, the cosine term in Eq. (1) is not a harmonic function but it can be, under certain specific restrictions, considered as a quasi-harmonic function, a very important fact in modeling the actual signals that contain displacement information. A more general signal contains higher order terms whose importance will be clarified later on in the paper. Eq. (2) corresponds to the case where the carrier is imaged by the optical system. There is an alternative expression for the phase: the carrier is not resolved by the optical system and only the modulation

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http://dx.doi.org/10.1016/j.optlaseng.2015.07.014
0143-8166/

function is captured. That is

$$\Psi(x) = \phi(x) \tag{3}$$

Eqs. (1)–(3) constitute the basic models utilized in fringe pattern analysis (see Chapter 10 of [4]). The image must contain a carrier to encode information. The carrier can be a deterministic signal in the case of moiré patterns, harmonic phase analysis (HARP) or a stochastic signal. Stochastic signals are utilized by speckle methods in all the different approaches, holography, speckle interferometry, speckle photography, white light speckles, digital image correlation (DIC). The stochastic signals lead to the formation of fringes by optical means or the equivalent of fringes by numerical correlation methods that replace the optical correlation.

The purpose of this paper is to utilize the Theory of Signal Processing to draw general conclusions on the retrieval and processing of displacement information and its derivatives. Therefore the concept of phase introduced in Eqs. (1)–(3) is a fundamental aspect in this paper. Although, for example, the DIC method does not overtly uses this concept, to get conclusions of general validity for the process of retrieval of displacement information and its derivatives it is necessary to cast this process in the more general context of the Theory of Signal Processing, where the concept of phase is a fundamental variable.

The phase of a fringe pattern can be obtained from a recorded image by the method of multiphase recording of signals, for example the four phases signal (see Chapter 10 of [4]). From the very beginning of the fringe pattern analysis of isothetic signals (e.g. moiré pattern fringes), an alternative was proposed, the utilization of the in-quadrature method [3]. However, this alternative, although at the time that it was proposed seemed to be fully equivalent to the multiple phase method, proved not to be so in practice when implemented through the FFT method that provides the in-phase and can also give in-quadrature signals. In this paper, it will be shown that both methods have a common root and properly implemented lead to the same result.

2. Frequency modulation model of isothetic fringes

The next step in the process of fringe pattern analysis is to introduce the concept of frequency modulation. The classical approach is followed in this section but it is done in terms relevant to fringe analysis, an essential point to understand spatial frequency modulation and its relationship to fundamental variables of Continuum Mechanics. In what follows are presented developments that are related to fringe pattern processing whether they are coming from a deterministic carrier or a stochastic carrier. The argument $\Psi(x)$ given in Eq. (2) can be expressed as follows:

$$[2\pi f_c x + \phi(x)] = \left(2\pi f_c x + 2\pi \int_0^x m(x)dx\right) \tag{4}$$

where we have made the substitution $\phi(x) = \int_0^x \frac{d\phi(x)}{dx}dx$ and called

$$\frac{d\phi(x)}{dx} = m(x) \tag{5}$$

In Eq. (5), $m(x)$ is the derivative of the modulation function, or instantaneous frequency of $\Psi(x)$. Fig. 1 illustrates an assumed cosinusoidal frequency rate of change of the modulation function. Estimates of the bandwidth of the signal that will appear in the power spectrum of the FT of $\Psi(x)$ can be made on the basis of the above assumption. Then

$$m(x) = \left[\frac{d\phi(x)}{dx}\right]_m \cos\frac{2\pi}{L_p}x \tag{6}$$

where the subscript m indicates the maximum value of the derivative in the considered interval L_p, Fig. 1.

We define

$$\left[\frac{d\phi(x)}{dx}\right]_m = \Delta f \tag{7}$$

The frequency of the modulation function signal is called $f_m = 1/L_p$, that is the inverse of one cycle of the modulation function. The frequency f_m corresponds to a given term of the expansion of the modulation function in a FT. Consequently, for this particular term, it can be written

$$[2\pi f_c x + \phi(x)] = \left(2\pi f_c x + 2\pi \int_0^x \Delta f \cos(2\pi f_m x)dx\right) \tag{8}$$

Upon integration one obtains

$$[2\pi f_c x + \phi(x)] = \left(2\pi f_c x + \frac{\Delta f}{f_m} \sin(2\pi f_m x)\right) \tag{9}$$

Then the signal containing the displacement information is of the form

$$S_m(x) = I_1 \cos(2\pi f_c x + \beta \sin(2\pi f_m x)) \tag{10}$$

where β is defined as the modulation index

$$\beta = \frac{\Delta f}{f_m} \tag{11}$$

The original sinusoidal carrier signal is not longer sinusoidal because to the carrier it is added a modulating term that transforms the signal from a sinusoid to a frequency modulated function that contains many harmonics. In order to evaluate the harmonics associated with the modulation function, Eq. (10) is expanded in a series of Bessel functions of the first kind and of order n of the argument β:

$$S_m(x) = I_1 \sum_{n=-\infty}^{n=+\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)]\ x \tag{12}$$

Utilizing the above expansion, a discretization of the actual distribution of harmonics is obtained; that is in place of one constant I_1 that concentrates all the energy of the sinusoidal carrier, a distribution of amplitudes is obtained. If one computes the FT of Eq. (12), the power spectrum of the signal is made of an infinite number of components of amplitude $I_{1\beta} = I_1 J_n(\beta)$ and frequencies, $f_n = f_c \pm nf_m$ (see Fig. 2).

It can be seen that the spectrum is concentrated around the carrier signal. Although the spectrum of the signal is infinite, from the plot of the Bessel functions of the first kind of order n (see Fig. 3) it appears that the extent of the spectrum of the modulated signal depends on the value of these arguments. Tabled values of Bessel function of order n are such that $|J_n(\beta) \to 0|$ as $n \to \infty$. Beyond a certain value of β the amplitude of the corresponding harmonic becomes negligible as illustrated in Fig. 3.

The above arguments lead to the Carson's rule [5] to estimate the bandwidth of a signal

$$BW = 2(1+\beta)f_m \tag{13}$$

In [5], the estimation of bandwidth was limited to $\beta < 2$. The larger the parameter β is, the largest will be the bandwidth of the signal. This is a very important fact for the process of data retrieval: small values of β lead to a narrow distribution that becomes closer to the model of Eq. (1) where the energy is concentrated in the I_1 constant and the signal is assumed to be a sinusoid.

3. Spectrum of isothetic signals

The next task is to relate the preceding developments to Continuum Mechanics variables. In the context of one dimensional signals, we can analyze the lines of equal projected displacement or isothetic lines (see Chapter 13 of Ref. [4]). Returning to the initial