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Generalized Method of Moment estimation of multivariate multifractal models

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ABSTRACT

Multifractal processes have recently been introduced as a new tool for modeling the stylized facts of financial markets and have been found to consistently provide certain gains in performance over basic volatility models for a broad range of assets and for various risk management purposes. Due to computational constraints, multivariate extensions of the baseline univariate multifractal framework are, however, still very sparse so far. In this paper, we introduce a parsimoniously designed multivariate multifractal model, and we implement its estimation via a Generalized Methods of Moments (GMM) algorithm. Monte Carlo studies show that the performance of this GMM estimator for bivariate and trivariate models is similar to GMM estimation for univariate multifractal models. An empirical application shows that the multivariate multifractal model improves upon the volatility forecasts of multivariate GARCH over medium to long forecast horizons.

1. Introduction

Multifractal (MF) processes have been recently introduced as a new tool for modeling the stylized facts of financial markets. In contrast to the additive structure of the seminal GARCH family of models, this new class of volatility models conceives volatility as a hierarchical, multiplicative process with heterogeneous components. The essential new feature of MF models is their ability of generating different degrees of long-term dependence in various powers of returns - a feature pervasively found in empirical financial data, cf. Lo (1991), Ding et al. (1993), Beran (1994), Lobato and Savin (1998), Zumbach (2004), among others. This feature also sets multifractal models apart from long memory models of the FIGARCH type that are unifractal by design. Research on multifractal models originated from statistical physics (Mandelbrot, 1974). Unfortunately, the models used in physics are of a combinatorial nature and suffer from non-stationarity due to their restriction to a bounded interval and the non-convergence of moments in the continuous-time limit. This major weakness of the early so-called multifractal model of asset returns (MMAR) proposed by Mandelbrot et al. (1997) has been overcome by the development of iterative versions of the multifractal approach in the econometrics literature, the Markov-switching multifractal model (MSM) proposed by Calvet and Fisher (2001, 2004) and the multifractal random walk proposed by Bacry et al. (2000). Various subsequent developments can be found, for example, in Lux (2008), Calvet et al. (2006) or Lux and Morales-Arias (2010). Lux and Segnon (2016) provide an up-to-date review of variants of multifractal models, available estimation techniques and empirical applications.

Although the multifractal model is a rather new tool in volatility modelling, various approaches have already been explored to estimate its parameters. The parameters of the combinatorial MMAR have been estimated via an adaptation of the scaling estimator and Legendre transformation approach from statistical physics although this approach has been shown to likely yield unreliable results for fat-tailed data subject to volatility clustering (the well-known stylized facts of financial data), cf. Lux (2004). Maximum likelihood (ML) estimation for Markov-switching multifractal models has been developed by Calvet and Fisher (2004), and Generalized Method of Moments (GMM) by Lux (2008).

So far, available multifractal models are mostly univariate ones and only a few authors have explored bivariate models c.f. Bacry et al. (2000), Calvet et al. (2006), Idier (2011), Liu (2008) and Liu and Lux (2014). However, for many important questions in empirical research, multivariate settings (exceeding bivariate) are preferable, cf. Bollerslev (1990), Liesenfeld and Richard (2003). For instance, the extension of GARCH models to mutivariate settings provides a number of different specifications although most of them are highly parameterized, for details cf. Bauwens et al. (2006) and Tsay (2006). In this paper, we present a very parsimonious multivariate multifractal model with only a minimum of parameters. In the bivariate case our model can be

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viewed as a special case of the more complex approach of Calvet et al. (2006) but it can be more easily extended to trivariate settings and beyond. Our main contribution in this paper is the derivation of a set of moment conditions that allows easy and fast estimation of this multivariate model.

The rest of this paper is organized as follows: Section 2 presents a brief review of multifractal models. Section 3 introduces a parsimonious multivariate multifractal model and details how it can be estimated via GMM. Monte Carlo simulations are conducted to assess the efficiency of the estimates. Section 4 provides an empirical application to a trivariate series of exchange rates. Concluding remarks are provided in Section 5. The Appendices A and B provides details of the analytical moment conditions.

2. Review of multifractal models

Mandelbrot et al. (1997) introduced the multifractal model of asset returns (MMAR) adapting his 1974 model of cascades of energy flux in statistical physics to the dynamics of financial volatility. In physics, these "cascades" are typically modeled by multiplicative operations on probability measures, cf. Mandelbrot (1974) and Harte (2001). However, in a time series context the combinatorial nature of MMAR appears unfortunate and with the non-causal nature of the time transformation from chronological to "business" time one also inherits non-stationarity of the resulting process due to the inherent restriction to a bounded interval.

These limitations have been overcome by the introduction of iterative versions of multifractal processes, the most seminal development being the Markov-switching multifractal model (MSM), cf. Calvet and Fisher (2001, 2004). In their approach, asset returns are modeled as:

$$r_t = \sigma\left(\prod_{i=1}^{k} M_t^{(i)}\right)^{1/2} \cdot u_t \tag{1}$$

with u_t drawn from a standard Normal distribution $N(0, 1)$ and instantaneous volatility being determined by the product of k volatility components or multipliers $M_t^{(1)}$, $M_t^{(2)}$..., $M_t^{(k)}$, and a constant scale parameter σ. Volatility components are renewed at time t with probability γ_i depending on their rank i within the hierarchy of multipliers or remain unchanged with probability $1 - \gamma_i$. The transition probabilities are specified by Calvet and Fisher (2001, 2004) in a specific form that guarantees consistency between the discrete-time MSM and a continuous-time limiting multifractal process built upon a hierarchy of Poisson processes of volatility components. Convergence of the discrete model to its continuous-time counterpart holds if transition probabilities are specified as:

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^{i-1})}, \quad \text{for } i = 1, 2, \ldots k, \tag{2}$$

with parameters $\gamma_1 \in [0, 1]$ and $b \in (1, \infty)$.

This iterative version of the multifractal model preserves the hierarchical structure of MMAR while dispensing with its restriction to a bounded interval. With Markovian structure, this model is completely "well-behaved" (i.e. it shares all the convenient properties of Markov-switching processes), and it is capable of capturing some important properties of financial time series, namely, volatility clustering and the power-law behaviour of the autocovariance function of absolute moments:

$$Cov(|r_t|^q, |r_{t+\tau}|^q) \propto \tau^{2d(q)-1}. \tag{3}$$

where the function $d(q)$ indicates that different powers q of absolute returns are characterized by different hyperbolic decay factors of their autocovariances. It is worthwhile to note, however, that the power-law behavior of the MSM model holds only approximately in a preasymptotic range. Rather than displaying asymptotic power-law behavior of autocovariance functions in the limit $t \to \infty$ or divergence of the spectral density at zero, the Markov-switching MF model is rather characterized by only 'apparent' long memory with an approximately hyperbolic decline of the autocorrelation of absolute powers over a finite horizon and exponential decline thereafter. In particular, approximately hyperbolic decline as expressed in Eq. (3) holds only over an interval $1 \ll \tau \gg b^k$ with b the parameter of the transition probabilities of Eq. (2) and k the number of hierarchical cascade levels.

3. Multivariate multifractal model

3.1. A parsimonious framework: volatility correlations without additional parameters

One of the common motivations of extending univariate asset pricing models to multivariate ones is modeling the co-movements of volatility of different assets. Unlike the additive structure of the volatility dynamics of GARCH and stochastic volatility models, multifractal models conceive volatility as a hierarchical product of heterogeneous components. This feature allows us to decompose volatility into a hierarchical multiplicative sequence of volatility components with different frequencies. The range of these components can stretch from higher frequencies (daily or even intra-daily) to more persistent ones reflecting prevailing long term macroeconomic or other factors which might jointly affect different assets to varying degrees. Such common or idiosyncratic factors will be captured by joint or isolated components within the hierarchy of volatility factors. This particular construction allows us to model volatility correlations among assets without the need to introduce new parameters that would be hard to estimate.[1] We can modulate the volatility correlations in this framework via the number of joint components. This is different from the approach of Calvet et al. (2006) who introduce two additional parameters capturing the probablity of joint arrivals of volatility innovations as well as the strength of volatility correlations within a bivariate MSM. While our model is nested as a special case in this more general approach, it has the advantage that it can easily be extended to higher order multivariate settings without having to cope with an increase in the number of parameters. This also distinguishes our approach from the multivariate multifractal random walk of Bacry et al. (2000) that comes with a full *nxn* matrix of additional parameters regulating the volatility dependence among n single time series.

Let us consider an N-dimensional process governing asset returns evolving in discrete time over the interval $[0, T]$ with equally spaced discrete time points $t = 1, \ldots, T$, and $r_t = (r_1, \ldots, r_N)'$:

$$r_t = \sigma. *[g(M_t)]^{1/2}. *u_t, \tag{4}$$

where σ, u_t are $N \times 1$ vectors and .* denotes element by element multiplication, u_t follows the multivariate standard Normal distribution with variance-covariance matrix Σ:

σ is a vector of constant scale parameters and can be viewed as unconditional standard deviation. $g(M_t)$ is a $N \times 1$ vector of the products of multifractal volatility components, i.e., $g(M_t) = [g(M_{1,t}), \ldots, g(M_{N,t})]'$:

$$g(M_q, t) = \prod_{i=1}^{j} M_{q,t}^{(i)}. * \prod_{l=j+1}^{k} M_{q,t}^{(l)}, M_{1,t}^{(i)} = M_{2,t}^{(i)} = \ldots M_{N,t}^{(i)}, \quad \text{for } 1 < i \leq j, \tag{5}$$

Eq. (5) states that each element $q = 1, \ldots, N$ of $g(M_t)$ is the instantaneous volatility of a univariate multifractal process. Within this framework, we introduce volatility co-movements in a parsimonious way without any additional parameters assuming that the N time series share a number of j joint cascades that govern the strength of their volatility correlations. Consequently, the larger j, the higher the correlation between them. Factors responsible for co-movements of the

[1] Our approach of allowing for different degrees of correlation at different frequencies is similar to studying such correlations via wavelet coherence analysis, cf. Ramsey (2002) for an introduction and Barunik et al. (2016) for a recent application.