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Weighted robust Basis Function for phase unwrapping

Oscar Dalmau [a,*], Mariano Rivera [a], Adonai Gonzalez [b]

[a] Centro de Investigacion en Matematicas A. C., Jalisco S/N Col. Valenciana, 36240 Guanajuato, Guanajuato, Mexico
[b] Centro de Investigaciones en Optica A. C., Loma del Bosque 115, 37150 Leon, Guanajuato, Mexico

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ABSTRACT

This work presents a robust algorithm for phase unwrapping. The proposed algorithm is based on the expansion of the estimated phase through a linear combination of a set of Basis Functions. We present a novel weighted robust functional which is minimised using a two step strategy. This model allows us to reduce the influence of noise and to remove inconsistent pixels in the estimation of the unwrapped phase. The proposed model assumes that the phase is smooth. Under this assumption, experiments demonstrate that if the phase is corrupted by high levels of noise, our model presents a better performance than state of the art algorithms. For low levels of noise, the results are comparable.

1. Introduction

The phase unwrapping process [1–6] is an important stage for interferometric data processing, Synthetic Aperture Radar (SAR) interferometry [7–14], Magnetic Resonance Imaging (MRI) [15], profilometry by fringe pattern projection [16], to mention just some applications. There are different approaches for phase unwrapping algorithms; some reviews on the topic are reported in [17–20] and references therein.

According to Itoh [1], the first differences of the unknown unwrapped phase ϕ are related to the wrapped first differences of the wrapped phase ψ by

$$\phi(a)-\phi(b)=\mathcal{W}(\psi(a)-\psi(b))+n, \qquad (1)$$

where a and b are neighbouring pixels, $\mathcal{W}$ is the wrapping operator, and n is a probable residual; see notation in Section 2.1. Then, the phase unwrapping process can be achieved by means of a least squares approach [21]; this is discussed in more detail in Section 2.1.

The main problem of least squares based formulations is that the residual n in (1) is not Gaussian, and as a consequence, the computed unwrapped phase may have large distortions [22]. In order to solve this problem, weighting techniques or robust formulations can be useful to achieve an accurate wrapped phase [20,22–24].

On the other hand, a Radial Basis Functions (RBF) expansion for phase unwrapping was recently proposed in [25]. An advantage of the RBF based unwrapping method is the implicit filtering of the unwrapped phases. This method produces accurate wrapped phases when the noise level is relatively low. However, the results are severely deteriorated for high levels of noise. In this work, we propose a new model for phase unwrapping that also approximates the reconstructed phase through a linear combination of a set of functions. Similar to Ref. [25], we approximate the unwrapped phase using RBF, although our proposal has two main differences with respect to [25]: first, we based our model on a robust potential [24,26], and second, we also introduced a novel weight to the robust functional, which allows us to control inconsistent pixels [22,19]. These two modifications lead to a weighted robust functional for phase unwrapping which is less sensitive to noise and is also able to automatically remove inconsistent pixels in the parameter estimation process. The experimental results show a better performance of the proposed model over the least-squares-based RBF model for phase unwrapping [25] in the case of high levels of noise. When the noise level is low, the results of our approach are comparable to results achieved by state of the art methods, including the least-squares-based RBF model [25]. Since our method is based on RBF, it performs an implicit noise filtering, opposite to the robust unwrapping method recently reported in [4,6,20].

The structure of the paper is as follows. In Section 2 we present an overview of the phase unwrapping problem. Section 3 presents our proposal with a detailed description of the new model. Section 4 presents numerical experiments that compare the results achieved by our proposal with state of the art algorithms. Finally, in Section 5, we present our conclusions.

2. Brief review of phase unwrapping

In this section, we briefly explain the phase unwrapping problem and introduce the notation used in this work. Next, we briefly review the RBF approach for phase unwrapping [25].

* Corresponding author.

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2.1. Phase unwrapping description

Let $(x,y) \in \mathcal{L}$ be a pixel in the lattice $\mathcal{L} = \{(x,y) | x = 1,2,\ldots,N_x; y = 1,2,\ldots,N_y\}$, where N_x and N_y are the numbers of rows and columns of the image, respectively. Then, given the wrapped phase $\psi(x,y)$, the problem consists in computing the unwrapped phase $\phi(x,y)$; the phases are related through the *wrapping operator* $\mathcal{W}(\cdot) \in (-\pi,\pi]$ in the following way:

$$\psi(x,y) = \mathcal{W}(\phi(x,y)) \stackrel{\text{def}}{=} \phi(x,y) + 2k(x,y)\pi, \tag{2}$$

$\forall (x,y) \in \mathcal{L}$, and $k(x,y) \in \mathbb{Z}$ is an integer field such that $\psi(x,y) \in (-\pi,\pi]$. In order to compute $\phi(x,y)$, one needs to invert the operator $\mathcal{W}$, which is equivalent to finding the integer field $k(\cdot,\cdot)$, which in turn is an ill-conditioned problem if no further information is added. Let us define the standard discrete derivatives and gradient as

$$\Delta_x \phi(x,y) = \phi(x+1,y) - \phi(x,y), \tag{3a}$$

$$\Delta_y \phi(x,y) = \phi(x,y+1) - \phi(x,y), \tag{3b}$$

$$\nabla \phi(x,y) = [\Delta_x \phi(x,y), \Delta_y \phi(x,y)]^T, \tag{3c}$$

and the wrapped discrete derivatives of $\psi(\cdot,\cdot)$ as follows:

$$\psi_x^{\omega}(x,y) \stackrel{\text{def}}{=} \mathcal{W}[\Delta_x \psi(x,y)], \tag{4a}$$

$$\psi_y^{\omega}(x,y) \stackrel{\text{def}}{=} \mathcal{W}[\Delta_y \psi(x,y)]. \tag{4b}$$

Assuming that $\exists (x,y)$, such that $|\nabla \phi(x,y)|_{\infty} > \pi$, one can estimate the unwrapped phase $\phi(x,y)$ using a standard least square formulation [22]. Since the least square procedure corresponds to the maximum likelihood estimation based on Gaussian noise (residuals), the classic least square algorithm for phase unwrapping is not reliable, since the wrapped phase is contaminated by impulsive noise produced by wrapping phase jumps larger than π [22].

2.2. Radial Basis Function for phase unwrapping

Ref. [25] presents an alternative to estimate the unwrapped phase $\phi(x,y)$ by means of linear combinations of RBF. They approximate the phase with a linear combination based on a single function (*i.e.*, RBF), and use the following expansion:

$$\phi(x,y) = \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} a_{ij} \phi_{ij}(x,y) = \boldsymbol{\phi}^T(x,y)\boldsymbol{a}, \tag{5}$$

where N_i and N_j correspond to the number of base functions in the 'x'- and 'y'-directions, respectively, and the base functions $\phi_{ij}(x,y) = \phi_i(x)\phi_j(y)$ are equally spaced in the domain of the image. In particular, in [25] the authors use Gaussian functions:

$$\phi_k(z) = \exp\left(-\frac{(z-\mu_k)^2}{2\gamma_k^2}\right), \quad \text{with } z \in \{x,y\} \tag{6}$$

where μ_k and γ_k are the position and the width of the Gaussian function, respectively; $\boldsymbol{\phi}(x,y)$ and $\boldsymbol{a}$ are the vectorisation[1] of the matrices $[\phi_{ij}(x,y)]$ and $[a_{ij}]$, respectively; *i.e.*,

$$\boldsymbol{\phi}(x,y) \stackrel{\text{def}}{=} \text{vec}\left([\phi_{ij}(x,y)]_{\substack{i=1,\ldots,N_i \\ j=1,\ldots,N_j}}\right). \tag{7}$$

If we define

$$\xi_k(z) \stackrel{\text{def}}{=} -\frac{z-\mu_k}{\gamma_k^2}, \tag{8}$$

and compute the gradient of $\phi(x,y)$ using the expansion (5), one obtains

$$\frac{\partial}{\partial x}\phi(x,y) = \sum_{i,j} a_{ij}\xi_i(x)\phi_{ij}(x,y) = \boldsymbol{\phi}_x^T(x,y)\boldsymbol{a}, \tag{9a}$$

$$\frac{\partial}{\partial y}\phi(x,y) = \sum_{i,j} a_{ij}\xi_j(y)\phi_{ij}(x,y) = \boldsymbol{\phi}_y^T(x,y)\boldsymbol{a}, \tag{9b}$$

where $\boldsymbol{\phi}_x(x,y)$ and $\boldsymbol{\phi}_y(x,y)$ are the vectorisation of the matrices $[\xi_i(x)\phi_{ij}(x,y)]$ and $[\xi_j(y)\phi_{ij}(x,y)]$, respectively; *i.e.*,

$$\boldsymbol{\phi}_x(x,y) \stackrel{\text{def}}{=} \text{vec}\left([\xi_i(x)\phi_{ij}(x,y)]_{\substack{i=1,\ldots,N_i \\ j=1,\ldots,N_j}}\right), \tag{10a}$$

$$\boldsymbol{\phi}_y(x,y) \stackrel{\text{def}}{=} \text{vec}\left([\xi_j(y)\phi_{ij}(x,y)]_{\substack{i=1,\ldots,N_i \\ j=1,\ldots,N_j}}\right). \tag{10b}$$

We can write the least square cost function for phase unwrapping [19,25] as follows:

$$U(\boldsymbol{a}) = \sum_{(x,y)} \left[\boldsymbol{\phi}_x^T(x,y)\boldsymbol{a} - \psi_x^{\omega}(x,y)\right]^2 + \left[\boldsymbol{\phi}_y^T(x,y)\boldsymbol{a} - \psi_y^{\omega}(x,y)\right]^2, \tag{11}$$

or its corresponding matrix formulation

$$U(\boldsymbol{a}) = \|\boldsymbol{\Phi}_x\boldsymbol{a} - \boldsymbol{\psi}_x^{\omega}\|_2^2 + \|\boldsymbol{\Phi}_y\boldsymbol{a} - \boldsymbol{\psi}_y^{\omega}\|_2^2. \tag{12}$$

Then, the minimisation of the last function can easily be obtained with the following closed formula:

$$\boldsymbol{a}^* = (\boldsymbol{\Phi}_x^T\boldsymbol{\Phi}_x + \boldsymbol{\Phi}_y^T\boldsymbol{\Phi}_y)^{-1}(\boldsymbol{\Phi}_x^T\boldsymbol{\psi}_x^{\omega} + \boldsymbol{\Phi}_y^T\boldsymbol{\psi}_y^{\omega}). \tag{13}$$

Thus, the estimation of the unwrapped phase is computed with

$$\hat{\phi}(x,y) = \boldsymbol{\phi}^T(x,y)\boldsymbol{a}^*, \quad \forall (x,y) \in \mathcal{L}. \tag{14}$$

This approach is based on a least square formulation. The method yields a good phase reconstruction when the noise level is low and the phase is smooth, see Section 4.2; however, the results deteriorate when the noise level increases, principally producing a reduced dynamic range of the estimated phase. In the next subsection, we present a robust formulation that diminishes this problem.

3. Weighted robust RBF for phase unwrapping

In order to solve the main problem of the RBF formulation for Phase Unwrapping, we present a weighted-robust formulation that tries to automatically remove the contribution of noise information in the estimation of $\boldsymbol{a}$ and also takes into account the dynamic range problem of the RBF model. The new functional is

$$U(\boldsymbol{a},s) = \sum_{(x,y)} \upsilon^2(x,y)\left(\rho(r_x(x,y)) + \rho(r_y(x,y))\right) + \lambda(s-1)^2, \tag{15}$$

where $\upsilon(x,y)$ is an inconsistency detector; *i.e.*, $\upsilon(x,y) = 1$ if (x,y) is an inconsistency site, and $\upsilon(x,y) = 0$ otherwise, $\rho(\cdot)$ is a robust function [26] and $r_x(x,y)$, $r_y(x,y)$ are residuals defined as follows:

$$r_x(x,y) = \boldsymbol{\phi}_x^T(x,y)\boldsymbol{a} - s\psi_x^{\omega}(x,y), \tag{16a}$$

$$r_y(x,y) = \boldsymbol{\phi}_y^T(x,y)\boldsymbol{a} - s\psi_y^{\omega}(x,y), \tag{16b}$$

where s is a scaling factor that allows correcting the dynamic range of the unwrapped phase, $\lambda > 0$ is a regularisation parameter, and finally, the last term of (15) promotes that the scaling factor be close to 1.

[1] The vectorisation of a matrix is a linear transformation which converts the matrix into a column vector.