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Comparison of two-dimensional integration methods for shape reconstruction from gradient data

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ABSTRACT

As a requisite and key step in some gradient-based measurement techniques, the reconstruction of the shape, more generally the scalar potential, from the measured gradient data has been studied for many years. In this work, three types of two-dimensional integration methods are compared under various conditions. The merits and drawbacks of each integration method are consequently revealed to provide suggestions in selection of a proper integration method for a particular application.

1. Introduction

In metrology, some physical quantities from measurement may not be our desired objective directly, but they may have certain relationship with our objective. The measurements are therefore still very useful and can be employed to get our desired quantities. Many optical metrology techniques belong to this indirect measurement. For instance, wavefront measurement techniques (Hartmann-based wavefront sensing [1–3], lateral shearing interferometry [4–6], *etc.*) reconstruct the wavefront from the slopes measured by optical sensors. Moreover, the technique of shape from shading [7–9] estimates the surface profile by integrating the calculated gradient data. Similarly, three-dimensional shape measurement for specular surfaces, *e.g.* phase measuring deflectometry [10–18], integrates gradient data from metrology to get the surface shape as shown in Fig. 1. All these techniques above only measure the derivatives of the wanted quantity. In order to achieve our final goal, a two-dimensional (2D) integration procedure is necessary to reconstruct the shape from the measured derivatives.

Due to its wide application, 2D integration methods are investigated by many researchers and there are lots of articles in literatures [7,19–29]. The 2D integration problem can be considered as solving a Poisson equation with Neumann boundary conditions [30]. Research on 2D integration methods can be found since 1970s for wavefront reconstruction [19,20,22,30]. Generally, finite difference approaches were employed in those methods to connect the measured slope and desired shape, and least squares estimations are made for shape reconstruction. Fourier transform has been introduced into 2D integration in 1980s [7,31]. By applying the properties of Fourier transform, the integration operation can be implemented easily and quickly due to the well-known Fast Fourier Transform (FFT) algorithm. In 2004, Li *et al.* [23] compared the finite difference based least squares integration method and Fourier transform integration method with showing the finite difference based least squares integration method has higher accuracy at that time. By considering the boundary conditions, Talmi and Ribak [24] pointed out the best solution of gradient integration could be expressed in a Fourier cosine series, not the periodic Fourier series. The cosine transform integration method is therefore proposed with providing the integration result at half-integer positions in 2006. Coming to 2008, Ettl *et al.* [25] introduced an integration method by employing the radial basis functions which is flexible and robust. In 2012, Bon *et al.* [26] proposed a boundary-artifact-free Fourier integration method by simply padding slope matrices with accordingly flipped and positive or negative slope values. By noticing the accuracy of the traditional finite difference based least squares integration method is limited by its biquadratic shape assumption, Huang and Asundi [27,28] proposed an iterative compensation approach to obtain more accurate integration results. Recently, Li *et al.* [29] improved the finite difference based least squares integration method by applying higher-order numerical differentiation formats.

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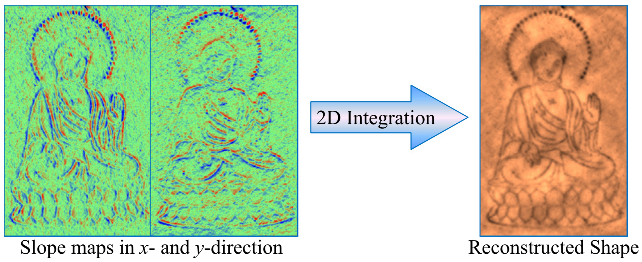

Fig. 1. The 2D integration is a vital process for the reconstruction of shape from the measured gradient data.

In this work, three families of integration methods are selected into our comparison, since they have been widely used in numerous applications. Several 2D integration methods are chosen as the representative of their corresponding families to make a comparison in both reconstruction accuracy and processing speed. These integration methods are invented in different application fields, and developed along their own paths. It is very interesting to know their merits and drawbacks after the improvement in recent years. Their "abilities" and "tempers" are revealed in order to help the selection of a proper 2D integration method for a specific application.

2. The 2D integration methods in comparison

There are three families of 2D integration methods to be compared in this work. The first family is the Finite-difference-based Least-squares Integration (FLI) methods. The second big family of integration methods is the Transform-based Integration (TI) methods. The third one is the Radial Basis Function based Integration (RBFI) method. Because we are interested in reconstruction of arbitrary shapes, the well-known modal wavefront reconstruction methods with the Zernike or other polynomials [32] are not compared here, which are more suitable for symmetrical optical components. During the writing up of this work, some other integration methods are noticed, such as the spline-based methods [33], showing convincing results and they may be strong potential competitors as well.

2.1. *Finite-difference-based least-squares integration methods with Southwell configuration*

The Traditional Finite-difference-based Least-squares Integration (TFLI) method in Southwell configuration [19] is well known and widely used not only due to its consistency between shape and slope locations (where you measure the slope, where you get the shape), but more significantly, because of its biquadratic spline shape in nature. The shape model in the Southwell configuration is essentially a biquadratic spline with an algorithm error of $O(h^3)$ where h stands for the interval of the sampling grids, whereas a bilinear curve in other similar configurations [20,22]. Due to its simple implementation and reliable performance, the TFLI method has been widely applied for shape reconstruction. The relations of the slope and shape in a M-by-N matrix are set locally in Eq. (1),

$$\begin{cases} \dfrac{z_{m,n+1}-z_{m,n}}{x_{m,n+1}-x_{m,n}} \triangleq \dfrac{s^x_{m,n+1}+s^x_{m,n}}{2}, & m=1,2,\dots,M, \\ & n=1,2,\dots,N-1 \\ \dfrac{z_{m,n+1}-z_{m,n}}{y_{m,n+1}-y_{m,n}} \triangleq \dfrac{s^y_{m+1,n}+s^y_{m,n}}{2}, & m=1,2,\dots,M-1, \\ & n=1,2,\dots,N \end{cases}, \tag{1}$$

where x, y, z are the world coordinates, "$\triangleq$" stands for "equal in the least squares sense", s^x and s^y denote the measured slope values in x- and y-directions, respectively. The subscripts m and n are the matrix indices. The estimation can be handled globally as presented in Eq. (2).

$$\begin{bmatrix} z_{1,1} \\ z_{2,1} \\ \vdots \\ z_{M,N} \end{bmatrix} = (\mathbf{D}^\mathbf{T}\mathbf{D})^{-1}\mathbf{D}^\mathbf{T}\mathbf{G}, \tag{2}$$

where $\mathbf{D}^\mathbf{T}$ stands for the transpose of $\mathbf{D}$. The matrix $\mathbf{D}$ (usually in sparse matrix format) and the vector $\mathbf{G}$ are

$$\mathbf{D} := \begin{bmatrix} \mathbf{D}^\mathbf{x} \\ \mathbf{D}^\mathbf{y} \end{bmatrix} = \left[\begin{array}{ccccccccc} -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ \hdashline -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{array}\right], \tag{3}$$

$$\mathbf{G} := \begin{bmatrix} \mathbf{G}^\mathbf{x} \\ \mathbf{G}^\mathbf{y} \end{bmatrix} = \frac{1}{2}\left[\begin{array}{c} (s^x_{1,2}+s^x_{1,1})(x_{1,2}-x_{1,1}) \\ (s^x_{1,3}+s^x_{1,2})(x_{1,3}-x_{1,2}) \\ \vdots \\ (s^x_{M,N}+s^x_{M,N-1})(x_{M,N}-x_{M,N-1}) \\ \hdashline (s^y_{2,1}+s^y_{1,1})(y_{2,1}-y_{1,1}) \\ (s^y_{3,1}+s^y_{2,1})(y_{3,1}-y_{2,1}) \\ \vdots \\ (s^y_{M,N}+s^y_{M-1,N})(y_{M,N}-y_{M-1,N}) \end{array}\right]. \tag{4}$$

With the development of metrology, an algorithm error of $O(h^3)$ at the integration stage may no longer be acceptable. Further investigation makes improvement on this path for a better accuracy during recent years. Iterations can be carried out to enhance the accuracy with Iterative Finite-difference-based Least-squares Integration (IFLI) method [27], which integrates the gradient residuals to implement iterative compensation onto the final result. Moreover, instead of using iterations, Li *et al.* [29] propose a more direct approach by considering higher order terms in Taylor expansion into the least squares estimation. Here we call it Higher-order Finite-difference-based Least-squares Integration (HFLI) method in this work. The expression of $\mathbf{G}$ is selected as Eq. (5) to maintain the same sparse matrix $\mathbf{D}$ as Eq. (3) which is usually the major concern of memory cost.

$$\mathbf{G} := \begin{bmatrix} \mathbf{G}^\mathbf{x} \\ \mathbf{G}^\mathbf{y} \end{bmatrix} = \frac{1}{24}\left[\begin{array}{c} 12(s^x_{1,2}+s^x_{1,1})(x_{1,2}-x_{1,1}) \\ (-s^x_{1,4}+13s^x_{1,3}+13s^x_{1,2}-s^x_{1,1})(x_{1,3}-x_{1,2}) \\ (-s^x_{1,5}+13s^x_{1,4}+13s^x_{1,3}-s^x_{1,2})(x_{1,4}-x_{1,3}) \\ \vdots \\ 12(s^x_{M,N}+s^x_{M,N-1})(x_{M,N}-x_{M,N-1}) \\ \hdashline 12(s^y_{2,1}+s^y_{1,1})(y_{2,1}-y_{1,1}) \\ (-s^y_{4,1}+13s^y_{3,1}+13s^y_{2,1}-s^y_{1,1})(y_{3,1}-y_{2,1}) \\ (-s^y_{5,1}+13s^y_{4,1}+13s^y_{3,1}-s^y_{2,1})(y_{4,1}-y_{3,1}) \\ \vdots \\ 12(s^y_{M,N}+s^y_{M-1,N})(y_{M,N}-y_{M-1,N}) \end{array}\right] \tag{5}$$

To disclose the improvement made in FLI family, a simple comparison between methods of TFLI, IFLI, and HFLI is carried out by integrating a gradient dataset with size of 256 (pixel) × 256 (pixel) × 2 (direction).

$$z(x,y) = 0.2 \times \Big\{ 3(1-x)^2 \exp[-x^2-(y+1)^2]. \\ -10\left(\frac{x}{5}-x^3-y^5\right)\exp(-x^2-y^2)$$