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Variational method for integrating radial gradient field

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ABSTRACT

We propose a variational method for integrating information obtained from circular fringe pattern. The proposed method is a suitable choice for objects with radial symmetry. First, we analyze the information contained in the fringe pattern captured by the experimental setup and then move to formulate the problem of recovering the wavefront using techniques from calculus of variations. The performance of the method is demonstrated by numerical experiments with both synthetic and real data.

1. Introduction

Non-contacting measurement is important in many areas, including medicine, on-line inspection, computer-aided design or manufacturing and reverse engineering. Traditionally, coordinate measurement machines have been used for 3-D mechanical part inspection. They are well established and widely accepted in industry but suffer from limitations such as high cost and low measurement speed. On the other hand, optical 3-D sensors have several advantages over tactile methods. For instance, non-contact and thus reduce the risk of induced deformations, fast response potential, full-field working principle, and high data resolution. Due to recent advances in computing technology, some of these techniques have become automated, easier to use in applications, and more efficient in data reduction. This has resulted in the development of full-field optical techniques that are being used for real-time profile measurements in a wide range of settings [1].

One of the simplest and most powerful methods to measure the emerging wavefront from a phase object is deflectometry. This technique is used for surface measurement where the local slopes of the surface are measured optically and the surface itself is reconstructed using an integration procedure. This approach has found several practical applications mainly due to its simplicity of operation.

The evaluation of optical elements is an active research field [2,3]. Recently, aspherical and freeform optical surfaces have attracted attention owing to their optical performance. Because of their design, these surfaces typically cannot be exactly measured using the same techniques employed with spherical surfaces, thus a suitable choice of the experimental setup and data processing methods are important for obtaining their accurate characterization. For this purpose, we present a method to integrate the obtained information from a deflectometry setup with circular fringe patterns. In the following sections, first we analyze the information contained in the fringe pattern captured by the experimental setup and then move to formulate the problem of recovering the wavefront using techniques from calculus of variations. The resulting algorithm performance is tested by two numerical wavefront reconstructions.

2. Principle of measurement and information processing

2.1. Experimental modeling

The experimental setup used in this work is based on the deflectometric technique proposed by Massig [4] and used on several research works [3,5,6]. In this setup, a CCD camera is focused on a computer monitor with the phase object situated between them. The monitor is used to display a fringe pattern, which is deformed by the phase object and the fringe pattern is imaging by the camera. In this kind of setup, parallel straight fringes are usually displayed on monitor. The fringe pattern captured by the camera is described by [3–5]

$$I_{\mathbf{x}} = a_{\mathbf{x}} + b_{\mathbf{x}} \cos\left(\frac{2\pi}{P}(\mathbf{x}\cdot\mathbf{v} + D\nabla\phi\cdot\mathbf{v})\right), \quad (1)$$

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where $\boldsymbol{x}=(x,y)$, $a_{\boldsymbol{x}}$ is the background illumination, $b_{\boldsymbol{x}}$ is the amplitude modulation, $\boldsymbol{v}=(\cos\varphi, \sin\varphi)$ is the normal direction vector of the pattern displayed on the screen, the term $\nabla\phi=(\partial\phi/\partial x, \partial\phi/\partial y)$ is the wavefront gradient, P is the pattern period, D is a constant related to the position of the lens and $\cdot$ denotes the dot product. The resultant measurement of this technique is the directional derivative of the wavefront ϕ oriented in the direction of $\boldsymbol{v}$. A common procedure to estimate the wavefront is to acquire two or more directional derivatives and integrate them; however, one drawback of this procedure is the computational time employed to acquire and integrate a large number of directional derivatives [7,8].

Although parallel straight fringes have been extensively used in optical metrology, a suitable choice to measure objects with radial symmetry is the use of circular fringe patterns. These patterns are capable of measuring in the x- and y-directions at the same time, *i.e.* this is equivalent to a radial derivative [9]. The circular fringe pattern deformed by the phase object is described by

$$I_{\boldsymbol{x}}=a_{\boldsymbol{x}}+b_{\boldsymbol{x}}\cos\left(\frac{2\pi}{P}(r+D\,\Phi_{\boldsymbol{x}})\right), \tag{2}$$

where $r=\sqrt{x^2+y^2}$ and $\Phi_{\boldsymbol{x}}$ represents the gradient information obtained from the phase object. It is important to remark that because the wavefront shifts the fringe pattern in the normal direction, this kind of fringe pattern is not sensitive to rotational changes on the gradient field; for a circular fringe pattern this means into the radial direction. As it was reported previously [9], the circular fringe patterns are capable of measurement of radial derivative, meaning that the gradient information obtained from the phase object can be expressed as

$$\frac{\partial\phi}{\partial r}=\Phi. \tag{3}$$

Moreover, consider a point $\boldsymbol{x}_1=(r_1\cos\theta_1, r_1\sin\theta_1)$ of the fringe pattern defined in Eq. (2); if this point is rotated by an angle α, the resultant point will be $\boldsymbol{x}_2=(r_1\cos(\theta_1+\alpha), r_1\sin(\theta_1+\alpha))$. As one can see, both points have the same radius, therefore they will have the same intensity value; that is, the fringe pattern will not undergo any change and consequently, any rotational movement cannot be detected; that is

$$\frac{\partial\phi}{\partial\theta}\approx 0. \tag{4}$$

2.2. Variational formulation

In the following, we shall describe how to formulate the problem of recovering the wavefront ϕ using techniques from calculus of variations. To this end, we propose the following minimization problem

$$\min_{\phi} E(\phi)\equiv\left\{\int_\Omega\left(\frac{\partial\phi}{\partial r}-\Phi\right)^2 d\boldsymbol{x}+\int_\Omega\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 d\boldsymbol{x}+\lambda R(\phi)\right\}, \tag{5}$$

where Φ is the gradient field obtained from the fringe pattern, $\Omega\subset R^2$ is the domain of integration, $R(\phi)$ is a regularization term to be selected and $\lambda>0$ a Lagrange multiplier. The first two functionals in $E(\phi)$ are fitting terms which come from our above discussion (Eqs. (3) and (4)), while the regularization term $R(\phi)$ is selected by using *a priori* information of the wavefront hence imposing properties to it.

By using the well-known formulas

$$\frac{\partial\phi}{\partial r}=\cos\theta\frac{\partial\phi}{\partial x}+\sin\theta\frac{\partial\phi}{\partial y}$$
$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\sin\theta\frac{\partial\phi}{\partial x}+\cos\theta\frac{\partial\phi}{\partial y},$$

it is possible to re-write the energy $E(\phi)$ in vectorial representation as follows:

$$E(\phi)=\int_\Omega(\boldsymbol{p}\cdot\nabla\phi-\Phi)^2 d\boldsymbol{x}+\int_\Omega(\boldsymbol{q}\cdot\nabla\phi)^2 d\boldsymbol{x}+\lambda\int_\Omega|\nabla\phi|^2 d\boldsymbol{x}, \tag{6}$$

where $\boldsymbol{p}=(\cos\theta, \sin\theta)$ is the orientation vector, $\boldsymbol{q}=(-\sin\theta, \cos\theta)$ and we have defined $R(\phi)$ as the L_2-norm of the gradient of ϕ. We selected this norm given the smooth properties of the wavefront to be recovered. Different regularization terms such as the Total Variation [10] of ϕ or even high-order ones, see Ref. [11] for an extensive review, may also be used depending upon the properties of ϕ.

In order to minimize Eq. (6), the first order optimality condition or Euler–Lagrange equation has to be derived. In the formal derivation we assume that the vector field ϕ is smooth enough such that gradients are well defined and the variation $\delta\phi$ has compact support over Ω so that we can use the divergence theorem to get rid of the boundary term.

Theorem 2.1. *Let $E(\phi)$ be defined as in (6). Then the firs variation is given by*

$$\frac{\partial E(\phi(\boldsymbol{x}))}{\partial\phi}=-\nabla\cdot\boldsymbol{w}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega,$$

with boundary conditions

$$\frac{\partial(\boldsymbol{p}\cdot\nabla\phi-\Phi)\boldsymbol{p}}{\partial\boldsymbol{\nu}}=0,\quad \frac{\partial(\boldsymbol{q}\cdot\nabla\phi)\boldsymbol{q}}{\partial\boldsymbol{\nu}}=0,\quad \text{and}\quad \frac{\partial\phi}{\partial\boldsymbol{\nu}}=0 \tag{7}$$

where $\boldsymbol{\nu}$ denotes the outer normal to the boundary and the flux field $\boldsymbol{w}$ is given by

$$\boldsymbol{w}=(\boldsymbol{p}\cdot\nabla\phi-\Phi)\boldsymbol{p}+(\boldsymbol{q}\cdot\nabla\phi)\boldsymbol{q}+\lambda\nabla\phi. \tag{8}$$

The proof of this Lemma is presented in the Appendix.

2.3. Numerical solution

First of all the following transformation must be considered:

$$y=r-r_o,\quad x=c-c_o,\quad \theta_{x,y}=\arctan_2(y,x)$$

where (r,c) is the position (row and columns) given by the field gradient matrix, (r_o,c_o) is the center of rotation and $\arctan_2$ is the tangent inverse function in the range $(-\pi,\pi]$.

Let $\phi_{i,j}=\phi(x,y)$ to denote the value of a grid function ϕ at point $(i,j)=(x,y)$ defined on the cell-centered grid

$$\Lambda_h=\{(x,y)\in\Lambda|(x,y)=((2i-1)h_x/2,(2j-1)h_y/2),\quad 1\le i\le m,\ 1\le j\le n\},$$

consisting of $m\times n$ cells of size $h_x\times h_y$ with $h_x=1/m, h_y=1/n$ the grid spacing. Note that $\Omega_h\subset\Lambda_h$ is the true region of integration.

To approximate the derivatives, we use forward and backward finite differences defined as follows:

$$\delta_x^{\pm}\phi_{i,j}=\pm(\phi_{i\pm1,j}-\phi_{i,j})/h_x,\quad \text{and}\quad \delta_y^{\pm}\phi_{i,j}=\pm(\phi_{i,j\pm1}-\phi_{i,j})/h_y.$$

Hence the numerical approximation of the Euler–Lagrange equation is

$$\nabla\cdot\boldsymbol{w}_{i,j}=\delta_x^- w_{1i,j}+\delta_y^- w_{2i,j}=0\quad \text{in } \Omega_h, \tag{9}$$

where

$$\boldsymbol{w}_{i,j}=-\psi_{i,j}(\cos\theta_{i,j}, \sin\theta_{i,j})^T-\vartheta_{i,j}(-\sin\theta_{i,j}, \cos\theta_{i,j})^T -(\delta_x^+\phi_{i,j},\delta_y^+\phi_{i,j})^T,$$

$$\psi_{i,j}=\cos\theta_{i,j}\delta_x^+\phi_{i,j}+\sin\theta_{i,j}\delta_y^+\phi_{i,j}-\Phi_{i,j}\quad \text{and}$$
$$\vartheta_{i,j}=-\sin\theta_{i,j}\delta_x^+\phi_{i,j}+\cos\theta_{i,j}\delta_y^+\phi_{i,j}.$$

By using backward derivatives to compute the divergence and doing some algebraic simplification, Eq. (9) can be written as a