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# On the computation of detection error probabilities under normality assumptions

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## HIGHLIGHTS

- This note presents a method for computing detection error probabilities in a closed form.
- The method is applicable to a particular class of log-consumption models with i.i.d. Gaussian errors.
- It provides the exact value of detection error probabilities and enables us to analytically show their properties unlike the existing simulation-based method.

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## ABSTRACT

This note describes a simple method for computing detection error probabilities under log-consumption models with i.i.d. Gaussian errors. The method is applicable to a class of models widely used in the literature, including the random walk, trend-stationary, long-run risk, and idiosyncratic risk models.


## 1. Introduction

Hansen and Sargent (2008a) and Barillas et al. (2009) used detection error probabilities to demonstrate that a moderate amount of concern about model misspecification under multiplier preferences can substitute for an implausibly high level of risk aversion. The computation of these detection error probabilities is under the assumption that the log consumption streams an agent faces in an endowment economy follow a random walk or trend-stationary process with i.i.d. Gaussian errors. The computational procedure relies entirely on simulation. In this note, we show that it is possible to compute the detection error probabilities using the cumulative distribution function under a class of models widely used in the literature, including the random walk, trend-stationary, long-run risk, and idiosyncratic risk models.

Under the random walk and trend-stationary models, Djeutem (2014) was the first to show that detection error probabilities can be calculated in a closed form. However, this note extends these results and makes the following unique contributions. First, it demonstrates that there are closed-form solutions for detection error probabilities if the value function is linear in i.i.d. Gaussian shocks, which also holds for a particular class of long-run and idiosyncratic risk models.[1] Thus, it provides a generalization of the formula in two directions.[2] Second, it presents a method for calculating standard errors of the overall detection error probability using the delta method.

The advantages of our result described here are twofold. The first is that it more quickly and easily provides the exact value of the detection error probabilities and enables us to test for their statistical significance unlike the existing simulation-based method. The second is that it enables us to reveal analytically their properties and therefore facilitates our interpretation. Our method, being based on a closed-form solution, is also useful if the overall detection error probability must be computed many

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1 The intuition for this is given in footnote 3 using a simple static setting.

2 Our proof differs from that of Djeutem (2014) in several respects and includes a correction of his proof.

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times, which holds for the asset-pricing applications in Hansen and Sargent (2008a) and Barillas et al. (2009).

This note is organized as follows. Section 2 briefly reviews the framework and computation procedure proposed by Hansen and Sargent (2008a) and Barillas et al. (2009). Section 3 explains our approach and shows how it is applicable to their asset-pricing applications. Section 4 discusses the extensions and some limitations of our formulas. All proofs are in the separate appendix.

## 2. The framework and computation procedure

Hansen and Sargent (2008a) and Barillas et al. (2009) used the finding that risk-sensitive preferences and multiplier preferences are observationally equivalent to reinterpret the quantitative finding of Tallarini Jr. (2000) concerning the risk aversion parameter. The risk-sensitive preferences are a special case of the recursive preferences suggested by Epstein and Zin (1989) and Weil (1990), in which the intertemporal elasticity of substitution is fixed at unity:

$$U_t = c_t - \beta\theta \ln\left(E_t\left[\exp\left(-\frac{U_{t+1}}{\theta}\right)\right]\right), \tag{1}$$

where $c_t$ is log consumption and $\beta \in (0, 1)$ is a discount factor. The parameter $\theta$ represents a measure of risk aversion

$$\theta = -\frac{1}{(1-\beta)(1-\gamma)}, \tag{2}$$

where $\gamma$ is a coefficient of relative risk aversion (RRA).

From the viewpoint of multiplier preferences, this parameter $\theta$ can be interpreted as the degree of an agent's concern about model misspecification. The detection error probabilities are used to quantify the degree to which the agent fears model misspecification. To illustrate the calibration method, let model A be an approximating model (a reference model), and let model B be a worst-case model associated with $\theta^{-1}$ (an alternative model in proximity to model A). Let $p_A$ denote the probability that a likelihood-ratio test selects model B when model A generates the data. Define $p_B$ similarly as the probability that selects model A when model B generates the data. Finally, define the overall detection error probability $p(\theta^{-1})$ by $p(\theta^{-1}) \equiv \frac{1}{2}(p_A + p_B)$.

In Hansen and Sargent (2008a) and Barillas et al. (2009), model A is assumed to be the following random walk and trend-stationary models

$$c_t = \mu + c_{t-1} + \sigma_\epsilon \epsilon_t, \tag{3}$$

$$c_t = \zeta + \mu t + z_t,\ \ z_t = \rho z_{t-1} + \sigma_\epsilon \epsilon_t,\ \ |\rho| < 1, \tag{4}$$

where $\epsilon_t \sim$ i.i.d.$N(0, 1)$. The corresponding worst-case model (model B) is then given by

$$c_t = \mu + \sigma_\epsilon w_{RW} + c_{t-1} + \sigma_\epsilon \epsilon_t,\ \ w_{RW} \equiv -\sigma_\epsilon/\theta(1-\beta), \tag{5}$$

$$c_t = \mu_1 + \mu_2 t + \sigma_\epsilon w_{TS} + \rho c_{t-1} + \sigma_\epsilon \epsilon_t,\ \ w_{TS} \equiv -\sigma_\epsilon/\theta(1-\rho\beta), \tag{6}$$

where $\mu_1 \equiv \zeta(1-\rho) + \rho\mu$ and $\mu_2 \equiv (1-\rho)\mu$. The procedure for calibrating the detection error probabilities developed by Hansen and Sargent (2008a) and Barillas et al. (2009) (henceforth, the BHS procedure) proceeds as follows.

1. Set the values of $\theta^{-1}$, $\beta$, $\zeta$, $\mu$, $\rho$, and $\sigma_\epsilon$. Simulate a path of length $T$ for $c_t$ using model A. Calculate the log-likelihood ratio, $\ln(L_A/L_B)$, to perform a test for distinguishing model A from model B. The test selects model A if $\ln(L_A/L_B) > 0$ and model B if $\ln(L_A/L_B) < 0$. Perform this test many times by simulating a large number of paths under model A, and count the fraction of $\ln(L_A/L_B) < 0$
$$p_A \equiv \text{Prob}\left(\ln\left(\frac{L_A}{L_B}\right) < 0\right) \approx \frac{\#\ln(L_A/L_B) < 0}{\#\text{simulations}}. \tag{7}$$
2. Simulate a large number of paths of length $T$ for $c_t$ using model B. Perform the log-likelihood ratio test, and count the fraction of $\ln(L_A/L_B) > 0$
$$p_B \equiv \text{Prob}\left(\ln\left(\frac{L_A}{L_B}\right) > 0\right) \approx \frac{\#\ln(L_A/L_B) > 0}{\#\text{simulations}}. \tag{8}$$
3. Calculate the overall detection error probability $p(\theta^{-1})$.
4. Repeat steps 1–3 for different values of $\theta^{-1}$ to obtain a graph of the overall detection error probability versus $\theta^{-1}$ (i.e., a detection error probability function).

The number of simulations for each computation of $p_A$ and $p_B$ is 100,000 or 500,000 in the BHS procedure (see Barillas et al. (2009, p. 2405) and Hansen and Sargent (2008a, p. 320)), so that the total number of simulations required is 200,000 or 1,000,000 to obtain *one* value of the overall detection error probability $p(\theta^{-1})$.

## 3. Simplification of the procedure

Let $\Phi(\cdot)$ be the standard normal cumulative distribution function. The following proposition states that we can compute $p(\theta^{-1})$ without relying on simulation under the random walk and trend-stationary models with i.i.d. Gaussian errors. To our knowledge, Djeutem (2014) has already noted this claim, but in a different context and form.

**Proposition 1.** *(i) For the random walk drift model, the detection error probabilities $p_A$ and $p_B$ are given by*

$$p_A = \Phi\left(-\frac{\sqrt{T}}{2}\frac{\sigma_\epsilon}{\theta(1-\beta)}\right) \ \ \text{and} \ \ p_B = 1 - \Phi\left(\frac{\sqrt{T}}{2}\frac{\sigma_\epsilon}{\theta(1-\beta)}\right). \tag{9}$$

*(ii) For the trend-stationary model, they are*

$$p_A = \Phi\left(-\frac{\sqrt{T}}{2}\frac{\sigma_\epsilon}{\theta(1-\rho\beta)}\right) \ \ \text{and} \ \ p_B = 1 - \Phi\left(\frac{\sqrt{T}}{2}\frac{\sigma_\epsilon}{\theta(1-\rho\beta)}\right). \tag{10}$$

*The overall detection error probability $p(\theta^{-1})$ is equal to $p_A$.*

A proof for this proposition is in Appendix A. In the proof, the key is that if the value function $U_t$ is *linear* in random shocks $\epsilon_t$, then a likelihood ratio $g(\epsilon_{t+1}) \equiv \hat{\pi}(\epsilon_{t+1})/\pi(\epsilon_{t+1})$ can be expressed as the exponential of a linear function of $\epsilon_{t+1}$. Here, $\pi(\epsilon_{t+1})$ is a conditional density of a sequence of random shocks $\{\epsilon_{t+1}\}$, and $\hat{\pi}(\epsilon_{t+1})$ is some other density in proximity to $\pi(\epsilon_{t+1})$ (i.e., a distorted density). By this result, the log-likelihood ratio $\ln(L_A/L_B)$ takes the familiar form under the AR(1) structure. Using this and the normality assumption of the shocks $\epsilon_t$, it is shown that the detection error probability $p_A$ in the BHS procedure represents the cumulative distribution function of a standard normal random variable (constructed from the i.i.d. Gaussian shocks $\epsilon_t$).[3] Given this result, the representation for $p_B$ follows from the symmetry of the standard normal distribution.

[3] The intuition of the proof is the following. To see the idea clearly, consider a simplified static structure. Note that the likelihood ratio $g(\epsilon)$ takes the form, $g(\epsilon) \equiv \hat{\pi}(\epsilon)/\pi(\epsilon) = \exp(-U/\theta)/E[\exp(-U/\theta)]$. Then the detection error probability $p_A$ is $p_A = \text{Prob}(\text{select model B}|\text{model A generated the data}) = \text{Prob}(\ln g^*(\epsilon) < 0|\pi(\epsilon))$, where $g^*(\epsilon) \equiv 1/g(\epsilon)$. (This inversion is merely for maintaining consistency with $L_A/L_B$ and is not essential.) If the value function $U$ is linear in $\epsilon$, say, $U = a_0 + a_1\epsilon$, then $p_A = \text{Prob}(\epsilon < -(\theta/a_1)\ln(E[\exp(-(a_1/\theta)\epsilon)]) |\pi(\epsilon)) = \text{Prob}(\epsilon < -a_1/2\theta|\pi(\epsilon))$, so that the distribution function $\Phi(\cdot)$ can be used because of $\epsilon \sim N(0, 1)$. Note that while this static-case derivation conveys our idea, our proof is needed in the dynamic setting that we treated.