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The bankruptcy problem in financial networks

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HIGHLIGHTS

- Bargaining Theory was previously applied to derive rules governing default of a single debtor having multiple creditors.
- Bankruptcy resolution within payments and interbank loan networks is more complex, due to multiple debtors owing multiple creditors.
- Bargaining Theory is extended to default resolution within such networks.
- Popular resolution rules, derived from Bargaining Theory for single debtor situations, are not similarly justified in financial networks.

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ABSTRACT

The *bankruptcy problem* of resolving a single debt owed to multiple creditors is extended to financial networks, where there are multiple debtors and creditors.

1. Introduction

The law and economics of default resolution in bankruptcy typically presume a single debtor that owes multiple creditors. But *multiple* debtors owe multiple creditors within interbank payment networks like Fedwire and CHIPS, as well as in decentralized networks created by complex interbank loan obligations. Nash Bargaining theory has previously been used to justify two popular rules for resolving default of a single debtor. I describe how to extend these rules to financial networks, but find that Nash Bargaining theory does not justify applying those rules to resolve defaults in those networks.

2. Default resolution rules

Dagan and Volij (1993) consider the following *bankruptcy problem*: one agent owes non-negative amounts denoted $c_1, c_2, \ldots, c_n$ to each of n creditors, but possesses an ability to pay only $E \leq \sum_{j=1}^{n} c_j$. The problem arises when the inequality is strict, in which case default is inevitable and must be resolved. A feasible *allocation* is a vector of default resolution payments $x_1, x_2, \ldots, x_n$ such that $\forall j, x_j \leq c_j$ and $\sum_j x_j = E$.

Dagan and Volij (op.cit.) study two default resolution rules :

(a) The *Proportional rule* allocation $x_1, x_2, \ldots, x_n$ requires the debtor to pay the *same* fraction $\lambda \leq 1$ in resolution to each creditor, i.e. $x_j = \lambda c_j$ where $\lambda \sum_j c_j = E$.

The Proportional rule has a very long history and is the basis for the extant resolution procedure in bankruptcy law (Kadens, 2010).

(b) The *Constrained Equality (CE) rule* allocates an equal *dollar* payment, denoted x, to each creditor, subject to the constraint that no one is paid more than originally *owed*.[1] That is, the CE rule finds an amount x and requires $x_j = \min(x, c_j)$ and $\sum_j [x_j = \min(x, c_j)] = E$.

[1] Levinthal's (1918) history of early bankruptcy law notes that ancient Jewish law required equal *dollar* (rather than percentage) payments to creditors, subject to the constraints that this would not compensate any creditors more than they were owed (op.cit, p. 234).

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Largely using their notation to facilitate comparisons, I generalize this framework to financial networks. In such networks, represent the payment c_{ij} *owed* by agent i to agent j by a nonnegative matrix:

$$\mathbf{C} = \begin{bmatrix} 0 & c_{12} & \dots & c_{1n} \\ c_{21} & 0 & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & 0 \end{bmatrix}. \quad (1)$$

Such networks include interbank lending, checks drawn on one bank that must be deposited in accounts at another bank, or payments owed as a result of mutual trading activities on an asset exchange. It is the nature of these networks that both c_{ij} and c_{ji} could be positive, and that there is no partition of indices into debtors and creditors.

A feasible network allocation is a matrix of resolution payments x_{ij} such that $\forall i, j \neq i : x_{ij} \leq c_{ij}$ and $\sum_{j \neq i} x_{ij} = E_i$, where E_i is agent i's ability to pay. Unlike the classical bankruptcy problem, here an agent's ability to pay is *endogenous* with the allocation, because any agent i's ability to pay includes payments it receives from other agents, i.e. $E_i \geq \sum_{j \neq i} x_{ij}$. In order to focus on the role of the interagent obligations matrix (1) rather than the exogenous component of the distribution of ability to pay, we assume strict equality, so that default resolution allocations must arise from redistributing obligations in (1), resulting in a matrix with each row total equal to its corresponding column total:

$$\mathbf{X} = \begin{bmatrix} 0 & x_{12} & \dots & x_{1n} \\ x_{21} & 0 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & 0 \end{bmatrix}. \quad (2)$$

We will use the following numerical example for illustration thro ughout:

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 30 & 0 & 20 & 20 \\ 10 & 30 & 0 & 10 \\ 10 & 0 & 20 & 0 \end{bmatrix}. \quad (3)$$

Inspecting (3), we see that agent #2 owes 70 in total but is owed only 30 in total. Hence it will have to default on some of these payments, triggering defaults by agents #3 and #4.

Elimam et al. (1996) and Eisenberg and Noe (2001) focused on a network default resolution rule in which defaulting agents are required to proportionally renege on all their respective creditors. This is the natural generalization of the classical proportional rule (a). We state this formally below:

(a^{Net}) The *Network Proportional rule* is a nonnegative matrix (2) requiring each agent i to pay a fraction $\lambda_i \leq 1$ in resolution of what it owes to each of the other agents, i.e.

$$x_{ij} = \lambda_i c_{ij} \text{ where } \sum_{j \neq i} x_{ij} = \lambda_i \sum_{j \neq i} c_{ij} = \sum_{j \neq i} x_{ji} = \sum_{j \neq i} \lambda_j c_{ji}. \quad (4)$$

The existence of a vector of fractions $(\lambda_1, \lambda_2, \dots, \lambda_n)$, that in conjunction with the obligations matrix $\mathbf{C}$ determines the Network Proportional rule allocation, was established by Eisenberg and Noe (op.cit.), who showed that one can be computed by solving a linear programming problem. A simplified exposition is given in Demange (2015), who solves the following linear program:

$$\begin{aligned} \boldsymbol{\lambda}^* &= \arg\max_{\lambda_1, \dots, \lambda_n} \sum_i \lambda_i \sum_{j \neq i} c_{ij} \\ s.t.\ \lambda_i \sum_{j \neq i} c_{ij} &- \sum_{j \neq i} \lambda_j c_{ji} \leq 0; \ \ i = 1, \dots, n. \end{aligned} \quad (5)$$

The constraints require that the solution fractions $\boldsymbol{\lambda}^*$ result in an aggregate of resolution payments *from* each agent i that does not exceed the aggregate of payments *to* it. The objective function is the aggregate of payments made throughout the network.

In our illustrative example (3), the numerical solution of (5) is $\boldsymbol{\lambda}^* = (1.0, 14.8\%, 34.4\%, 21.3\%)$. The associated *default resolution payments* x_{ij} are:

$$\mathbf{X}^P = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 4.426 & 0 & 2.951 & 2.951 \\ 3.443 & 10.328 & 0 & 3.443 \\ 2.131 & 0 & 4.262 & 0 \end{bmatrix}. \quad (6)$$

The natural network extension of the CE rule (b) is the following *Network CE rule:*

(b^{Net}) The *Network CE rule* is a nonnegative matrix (2) requiring each agent i to pay the same dollar amount x_i to each of its creditors, subject to the constraint that none of its creditors is paid more than originally *owed*. That is, the rule is a vector $x_1, x_2, \dots, x_n$ and requires that

$$\begin{aligned} &x_{ij} = \min(x_i, c_{ij}) \text{ and} \\ &\sum_{j \neq i} [x_{ij} = \min(x_i, c_{ij})] = E_i = \sum_{j \neq i} [x_{ji} = \min(x_j, c_{ji})]. \end{aligned} \quad (7)$$

Using (3), it is easy to verify that the vector $(x_1 = 10, x_2 = 5/3, x_3 = 5, x_4 = 10/3)$ enables feasible allocation of the following Network CE rule resolution payments:

$$\mathbf{X}^{\mathbf{CE}} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 5/3 & 0 & 5/3 & 5/3 \\ 5 & 5 & 0 & 5 \\ 10/3 & 0 & 10/3 & 0 \end{bmatrix}. \quad (8)$$

Using (7), denote $\sum_{j \neq i} x_{ij} \equiv l_i(\overrightarrow{x})$, where $l_i(\overrightarrow{x})$ denotes the total resolution payments *made* (i.e. liabilities) by agent i, and $\sum_{j \neq i} x_{ji} = a_i(\overrightarrow{x})$, the total resolution payments *received* (i.e. assets) by agent i. We prove the following constructive existence proposition below:

Proposition 1. *Under the Network* CE *rule* (7), *the feasible allocation set is nonempty. Moreover, the maximal* $\overrightarrow{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ *defining its required payments* $x_{ij}^* = min(c_{ij}, x_i^*)$ *can be found by solving*

$$\begin{aligned} \max_{X_{ij}} \sum_i \sum_{j \neq i} x_{ij} &\equiv \max_{x_1, \dots, x_n} \sum_i \sum_{j \neq i} min(x_i, c_{ij})\, s.t. \\ l_i(\overrightarrow{x}) &\equiv \sum_{j \neq i} min(x_i, c_{ij}) = a_i(\overrightarrow{x}) \\ &\equiv \sum_{j \neq i} min(x_j, c_{ji}), \ \ i = 1, \dots, n. \end{aligned} \quad (9)$$

Proof. Define the vector-valued maps $\mathbf{l}(\overrightarrow{x}) \equiv (l_1(\overrightarrow{x}), \dots, l_n(\overrightarrow{x}))$ and $\mathbf{a}(\overrightarrow{x}) \equiv (a_1(\overrightarrow{x}), \dots, a_n(\overrightarrow{x}))$ on the subset S of vectors $\overrightarrow{x}$ in which $x_i \in [0, \max_{j \neq i} c_{ij}], i = 1, \dots, n$. This subset is a complete lattice with the usual ordering $\leq$ on n-vectors. $\mathbf{l}(\overrightarrow{x})$ is monotone increasing on S, and hence has an inverse. Because $\mathbf{a}(\overrightarrow{x})$ is monotone nondecreasing on S, the map $\mathbf{f} : S \to S; f(\overrightarrow{x}) = \mathbf{l}^{-1}(\mathbf{a}(\overrightarrow{x}))$ is monotone on the complete lattice S. A fixed point of $\mathbf{f}$ satisfies the constraints in (9). By the Knaster–Tarski Fixed Point Theorem,[2] the map has a set of fixed points which is also a complete lattice (and hence nonempty), and hence has a maximal element. So it can be found by solving (9). □

The matrix (8) was found by substituting (3) into problem (9) and numerically solving it.

[2] See https://en.wikipedia.org/wiki/Knaster%E2%80%93Tarski_theorem.