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A note on low-dimensional Kalman smoothers for systems with lagged states in the measurement equation

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HIGHLIGHTS

- We derive a modified Kalman smoother for state space systems with lagged states in the measurement equation.
- Computationally efficient algorithms for the modified Kalman smoother are presented.
- It is proven that a conjecture in Nimark (2015) for obtaining a modified Kalman smoother is in general not correct.

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ABSTRACT

In this paper we derive a modified Kalman smoother for state space systems with lagged states in the measurement equation. This modified Kalman smoother minimizes the mean squared error (MSE). Computationally efficient algorithms that can be used to implement it in practice are discussed. We also show that the conjecture in Nimark (2015) that the output of his modified Kalman filter for this type of systems can be plugged into the standard Kalman smoother is in general not correct. The competing smoothers are compared with regards to the MSE.

In this note, we consider state space systems with a lagged state in the measurement equation for which Nimark (2015) derives a modified low-dimensional Kalman filter. Nimark (2015) also states, without a formal derivation, that the filtered state estimates from the modified filter can be plugged into the standard, i.e., unmodified, Kalman smoother of Hamilton (1994). In this paper we show that to use the filtered state estimates from the modified filter, we also need to modify the Kalman smoother to obtain the MSE-minimizing smoothed state estimates. That is, the claim that the filtered estimates from Nimark's (2015) modified filter can be plugged into the standard Kalman smoother is in general not correct. In what follows, we derive three modified Kalman smoothers that all can be used in combination with the modified Kalman filter of Nimark (2015). The first is based on the same principles as the one in Hamilton (1994). The second and third, and computationally more efficient, smoothers are a modified version of the smoother of de Jong (1988, 1989) and Kohn and Ansley (1989), and a modified version of the disturbance-smoother-based state smoother of Koopman (1993). Finally, the minimum variance estimator for the smoothed states will be compared to the Nimark (2015) smoother.

1. The state space model

In this note we stick as close as possible to the notation of Nimark (2015) and consider the state space model

$$X_t = AX_{t-1} + Cu_t, \qquad Z_t = D_1X_t + D_2X_{t-1} + Ru_t, \tag{1.1}$$

where u_t is a m-dimensional vector of disturbances being multivariate normally distributed with zero mean and the identity as variance–covariance matrix. The observable at time t, Z_t, is a $p \times 1$ vector and the state vector X_t is of dimension $n \times 1$. Similar to Nimark (2015), we use for the conditional expectation and variance the notations

$$X_{t|t-s} = \mathbb{E}(X_t|Z_{1:t-s}, X_{0|0}), \qquad P_{t|t} = \mathbb{E}((X_t - X_{t|t})(X_t - X_{t|t})'),$$

with $Z_{1:t} = (Z_1', \ldots, Z_t')'$ and we initialize the system by $X_0 \sim N(X_{0|0}, P_{0|0})$.

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2. The modified Kalman filter

The standard solution to apply the Kalman filter to the state space system (1.1) is obtained by augmenting the state vector with lagged states. A modified Kalman filter, which operates with an n-dimensional state vector, was derived by Nimark (2015). Nimark's (2015) modified Kalman filter can be summarized by the following recursion

$$\tilde{Z}_t = Z_t - \tilde{D}X_{t-1|t-1}, \qquad P_{t|t-1} = AP_{t-1|t-1}A' + CC', \tag{2.1}$$

$$X_{t|t} = AX_{t-1|t-1} + K_t\tilde{Z}_t, \qquad P_{t|t} = P_{t|t-1} - K_tF_tK_t', \tag{2.2}$$

with $\tilde{D} = (D_1A + D_2)$ and where the Kalman gain is given by $K_t = U_tF_t^{-1}$ with

$$U_t = \mathbb{E}(X_t\tilde{Z}_t') = AP_{t-1|t-1}\tilde{D}' + CC'D_1' + CR', \tag{2.3}$$

$$F_t = \mathbb{E}(\tilde{Z}_t\tilde{Z}_t') = \tilde{D}P_{t-1|t-1}\tilde{D}' + (D_1C + R)(D_1C + R)'. \tag{2.4}$$

3. On the Kalman smoother for systems with a lagged state in the measurement equation

To derive the updating equations which are purely based on filtered states and not on the observables, Hamilton (1994) uses the following approach.[1] By the formula for updating linear projections (Eq. [4.5.30] in Hamilton, 1994) one gets

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0}) = X_{t|t} + \hat{J}_t(X_{t+1} - X_{t+1|t}),$$

with $\hat{J}_t = P_{t|t}A'P_{t+1|t}^{-1}$. In a next step, Hamilton (1994) argues that $\mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0})$ is equal to $\mathbb{E}(X_t|X_{t+1}, Z_{1:T}, X_{0|0})$, as the error

$$X_t - \mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0})$$

is uncorrelated with Z_{t+j}, for $0 < j \leq T - t$. While this is true for a standard Kalman filter, as shown in Hamilton (1994), this is (in general) not the case for state space systems with a lagged state in the measurement equation, i.e., in general for state space systems of the form (1.1)

$$\mathrm{Corr}(X_t - \mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0}), Z_{t+1}) \neq 0$$

and therefore

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0}) \neq \mathbb{E}(X_t|X_{t+1}, Z_{1:T}, X_{0|0}). \tag{3.1}$$

As a consequence, the smoother stated in Eq. (4.2) in Nimark (2015)[2]

$$\hat{X}_{t|T} = X_{t|t} + \hat{J}_t(X_{t+1|T} - X_{t+1|t}), \qquad \hat{J}_t = P_{t|t}A'P_{t+1|t}^{-1}, \tag{3.2}$$

is in general not equal to $\mathbb{E}(X_t|Z_{1:T}, X_{0|0})$ as claimed by Nimark (2015). Note that in general the smoothed estimate, $\hat{X}_{t|T}$, (Eq. (3.2)) is also not minimizing the MSE to X_t conditional on the complete history of the observables $Z_{1:T}$.

This can be easily verified, e.g., by considering the special case $A = 0_{n\times n}$. Then, by (3.2), we get

$$\hat{X}_{T-1|T} = X_{T-1|T-1} \quad \Rightarrow \quad \mathrm{Var}(X_{T-1} - \hat{X}_{T-1|T}) = P_{T-1|T-1} \tag{3.3}$$

and in contrast for

$$X_{T-1|T} = X_{T-1|T-1} + P_{T-1|T-1}D_2'F_T^{-1}\tilde{Z}_T \tag{3.4}$$

we obtain

$$\mathrm{Var}(X_{T-1} - X_{T-1|T}) = P_{T-1|T-1} - P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1}. \tag{3.5}$$

[1] This state smoothing algorithm goes back to Anderson and Moore (1979) and Rauch et al. (1965).

[2] Note that there is a typo in Eq. (4.2) in Nimark (2015), where the index of $\hat{J}$ was $t-1$ instead of t, as in Hamilton (1994).

Both smoothers, (3.2) and (3.4), are obviously unbiased and as $P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1}$ is positive semidefinite if F_T is positive semidefinite it follows with (3.3) and (3.5)

$$\begin{aligned}\mathrm{MSE}(X_{T-1|T}) &= \mathrm{tr}(P_{T-1|T-1}) - \mathrm{tr}(P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1})\\ &\leq \mathrm{tr}(P_{T-1|T-1}) = \mathrm{MSE}(\hat{X}_{T-1|T}),\end{aligned}$$

i.e., the smoother, $\hat{X}_{T-1|T}$, is not the MSE-minimizing estimator of X_{T-1} given the complete history of the observables $Z_{1:T}$.

4. Kalman smoothing algorithms for the modified system

Similar to Hamilton (1994), the MSE-minimizing smoother for the modified system can be obtained using the updating equation for linear projections but with an adaption for systems with a lagged state in the measurement equation. Start by considering the conditional expectation $\mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0})$ and by applying the formula for updating a linear projection (Eq. [4.5.30] in Hamilton, 1994)

$$\begin{aligned}&\mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0})\\ &\quad = X_{t|t+1} + \mathbb{E}((X_t - X_{t|t+1})(X_{t+1} - X_{t+1|t+1})')\\ &\qquad \cdot \mathbb{E}((X_{t+1} - X_{t+1|t+1})(X_{t+1} - X_{t+1|t+1})')^{-1}(X_{t+1} - X_{t+1|t+1})\\ &\quad = X_{t|t+1} + P_{t+1,t|t+1}'P_{t+1|t+1}^{-1}(X_{t+1} - X_{t+1|t+1}),\end{aligned}$$

where $P_{t+1,t|t+1} = \mathbb{E}((X_{t+1} - X_{t+1|t+1})(X_t - X_{t|t+1})') = AP_{t|t} - K_{t+1}\tilde{D}P_{t|t}$. From the standard theory on state smoothing (see, e.g., Durbin and Koopman, 2012), we get the one-step ahead smoothed state as

$$X_{t|t+1} = X_{t|t} + P_{t|t}\tilde{D}'F_{t+1}^{-1}\tilde{Z}_{t+1}.$$

Future observables, Z_{t+j}, for $1 < j \leq T - t$, can be written as

$$\begin{aligned}Z_{t+j} &= \tilde{D}X_{t+j-1} + (D_1C + R)u_{t+j}\\ &= \tilde{D}\left(A^{j-2}X_{t+1} + \sum_{i=2}^{j-1}A^{j-1-i}Cu_{t+i}\right) + (D_1C + R)u_{t+j},\end{aligned}$$

where we use the notational convention that A^0 is the identity and A^n denotes the n-th power of the square matrix A. Therefore, using the same reasoning as in Hamilton (1994), we see that the prediction error

$$\begin{aligned}&X_t - \mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0})\\ &\quad = X_t - X_{t|t+1} - P_{t+1,t|t+1}'P_{t+1|t+1}^{-1}(X_{t+1} - X_{t+1|t+1})\end{aligned} \tag{4.1}$$

is uncorrelated with Z_{t+j} for $1 < j \leq T - t$. This follows because the prediction error (4.1) is by construction uncorrelated with X_{t+1}, and by assumption uncorrelated with $u_{t+j}, u_{t+j-1}, \ldots, u_{t+2}$. As a consequence, we get

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:T}, X_{0|0}) = \mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0}) \tag{4.2}$$

and by applying the law of iterated projections, as Hamilton (1994), we obtain the smoothed estimate, $\mathbb{E}(X_t|Z_{1:T}, X_{0|0})$, by projecting (4.2) on $Z_{1:T}$. The smoothed estimate is given by

$$X_{t|T} = \mathbb{E}(X_t|Z_{1:T}, X_{0|0}) = X_{t|t+1} + J_t(X_{t+1|T} - X_{t+1|t+1}), \tag{4.3}$$

with $J_t = P_{t+1,t|t+1}'P_{t+1|t+1}^{-1}$.

4.1. MSE of the smoothed state

Analogously to Hamilton (1994), by subtracting X_t from Eq. (4.3) and rearranging, we obtain

$$X_t - X_{t|T} + J_tX_{t+1|T} = X_t - X_{t|t+1} + J_tX_{t+1|t+1}. \tag{4.4}$$