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The power of the largest player

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HIGHLIGHTS

- Deviations between different influence measures for decisions in committees.
- The power of the largest player is treated significantly different by power indices.
- Implications for the design of decision rules —"inverse power index problem".
- Further criteria for the application dependent choice of a suitable power index.

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ABSTRACT

Decisions in a shareholder meeting or a legislative committee are often modeled as a weighted game. Influence of a member is then measured by a power index. A large variety of different indices has been introduced in the literature. This paper analyzes how power indices differ with respect to the largest possible power of a non-dictatorial player. It turns out that the considered set of power indices can be partitioned into two classes. This may serve as another indication which index to use in a given application.

1. Introduction

Consider a community association with four property owners having shares of 50%, 26%, 15%, and 9%, respectively. Assume that decisions are of a simple "yes" or "no" nature and that the owners decide with a two-thirds majority rule. Such a decision environment can be modeled as a weighted game, where the players have non-negative weights $w_1, \ldots, w_n$. Any subset S of the players, called coalition, can adopt a proposal if and only if the sum of their weights $\sum_{i \in S} w_i$ meets or exceeds a given positive quota q. The collection $[q; w_1, \ldots, w_n]$ is then called a weighted game, e.g., [0.67; 0.50, 0.26, 0.15, 0.09] in our example. Note that those voting weights are often a poor proxy for players' influence. Whenever S is a coalition including the third but excluding the fourth player and T is the coalition obtained from exchanging player three by player four, then coalition S can bring through a proposal if and only if coalition T can do. So, the third and the fourth player are symmetric in terms of their influence on the decision, which is not reflected by the weights.

The literature has thus introduced several more sophisticated ways of measuring a players' influence in weighted games. Unfortunately, different indices can lead to very different predictions. For our example we obtain relative power distributions of $(0.50, 0.30, 0.10, 0.10)$, $\frac{1}{12} \cdot (7, 3, 1, 1)$, $(1, 0, 0, 0)$, or $(0.40, 0.20, 0.20, 0.20)$ for the Penrose–Banzhaf index, the Shapley–Shubik index, the nucleolus, and the Public Good index, respectively.

One way to decide which power index to choose for a given application is to employ one of the known axiomatizations, see e.g. Dubey (1975) and (Dubey and Shapley, 1979), and to check which axioms are satisfied. Here, we consider the power of the largest player, without full power. It will turn out that the possible values differ significantly for different power indices, which may also allow to exclude the suitability of certain power indices in a given application. Although this theoretical question is quite natural, it has not been treated in the literature so far.

Another application of our results stems from the so-called *inverse power index problem*, see e.g. De et al. (2017); Koriyama et al. (2013); Kurz (2012); Kurz et al. (2017). It asks for a simple or weighted game v such that the corresponding power distribution (according to a given power index p) meets a given ideal power distribution σ as closely as possible. Since there is only a finite number of different weighted or simple games, it is obvious that

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some power vectors cannot be approximated too closely if the number of voters is small. (Alon and Edelman, 2010) show that there are also vectors that are hard to approximate by the Penrose–Banzhaf index of a simple game if most of the mass of the vector is concentrated on a small number of coordinates. Generalizations and impossibility results for other power indices have been obtained in Kurz (2016). So, if we know that $p_i(v) = 1$ or $p_i(v) \le \lambda$, for any simple game v, and σ_i lies somewhere in the middle of the interval $[\lambda, 1]$, then $p(v)$ has a significant distance to σ provided that λ is not close to 1.

2. Preliminaries

By $N = \{1, \dots, n\}$ we denote the set of players. A *simple game* is a surjective and monotone mapping $v : 2^N \to \{0, 1\}$ from the set of subsets of N into a binary output $\{0, 1\}$. *Monotone* means $v(S) \le v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. The values of this mapping can be interpreted as follows. For each subset S of N, called *coalition*, we have $v(S) = 1$ if the members of S can adopt a proposal even though the members of $N \setminus S$ are against it. If $v(S) = 1$ we speak of a *winning coalition* and a *losing coalition* otherwise. A winning coalition S is called *minimal* if all of its proper subsets are losing. Similarly, a losing coalition T is *maximal* if all of its proper supersets are winning. A simple game v is weighted if there exist weights $w_1, \dots, w_n \in \mathbb{R}_{\ge 0}$ and a quota $q \in \mathbb{R}_{>0}$ such that $v(S) = 1$ exactly if $w(S) := \sum_{i \in S} w_i \ge q$. Two players i and j are called *symmetric*, in a given simple game v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $\emptyset \subseteq S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* if $v(S) = v(S \cup \{i\})$ for all $\emptyset \subseteq S \subseteq N \setminus \{i\}$, i.e., player i is not contained in any minimal winning coalition. A player that is contained in every minimal winning coalition is called a veto player. If $\{i\}$ is a winning coalition (note that $\emptyset$ is a losing coalition), then player i is a *passer*. If additionally all other players are null players, then we call player i a *dictator*.

A power index p is a mapping from the set of simple (or weighted) games on n players into $\mathbb{R}^n$. By $p_i(v)$ we denote the ith component of $p(v)$, i.e., the power of player i. As an example consider the *Shapley–Shubik index*:

$$\mathrm{SSI}_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot (v(S \cup \{i\}) - v(S)).$$

The list of power indices that have been proposed in the literature so far is long. In order to keep the paper compact and self-contained, we follow the proposed taxonomy of Kurz (2016) and refer the reader, e.g., to that paper for more references and details. We call p *positive* if $p(v) \in \mathbb{R}^n_{\ge 0} \setminus \{0\}$ and *efficient* if $\sum_{i=1}^n p_i(v) = 1$ for all games v. For any positive power index p we obtain an efficient version by $p_i(v) / \sum_{j=1}^n p_j(v)$. Applying this to $\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$ gives the *Penrose–Banzhaf index* BZI. We call a coalition $S \cup \{i\}$ *critical* for i, if $v(S \cup \{i\}) - v(S) = 1$. Then player i called *critical*. Note that not all players of a critical coalition are critical.

Instead of counting critical coalitions, we can also count the minimal winning coalitions containing a given player i. Normalizing to an efficient version, as above, gives the *Public Good index* PGI. The so-called *equal division counting function* gives each relevant player of a counted coalition the same share, so that they sum up to one. More concretely

$$\sum_{\{i\} \subseteq S \subseteq N \,:\, S \text{ is minimal winning}} \frac{1}{|S|}$$

gives the non-normalized version of the Deegan–Packel index DP for player i, i.e., it arises from the PGI by equal division. Equal sharing among the critical players of a coalition turns the Penrose–Banzhaf index into the Johnston index Js. The definition of the *nucleolus* Nuc is a bit more involved. For a simple game v and a vector $x \in \mathbb{R}^n$ we call $e(S, x) = v(S) - x(S)$ the *excess* of S at x, where $x(S) := \sum_{i \in S} x_i$. It can be interpreted as quantifying the coalition's dissatisfaction and potential opposition to an agreement on allocation x. For any fixed x let $S_1, \dots, S_{2^n}$ be an ordering of all coalitions such that the excesses at x are weakly decreasing, and denote these ordered excesses by $E(x) = \left(e(S_k, x)\right)_{k=1,\dots,2^n}$. Vector x is *lexicographically less* than vector y if $E_k(x) < E_k(y)$ for the smallest component k with $E_k(x) \ne E_k(y)$. The *nucleolus* $x^\star$ of v is then uniquely defined as the lexicographically minimal vector x with $x(N) \le v(N) = 1$, cf. (Schmeidler, 1969). For simple games we automatically have $x^\star \in \mathbb{R}^n_{\ge 0}$ and $x^\star(N) = 1$. Several authors restrict the definition to imputations, where $x^\star_i \ge v(\{i\})$.

We call a power index p *symmetric* if $p_i(v) = p_j(v)$ for symmetric players i, j in v. If $p_i(v) = 0$ for every null player i of v, then we say that p satisfies the *null player property*. The six power indices introduced so far are positive, efficient, symmetric, satisfy the null player property and are defined for all simple games.

There are a few other power indices that are just defined for a weighted game v and based on representations. For our initial example we have $[0.67; 0.50, 0.26, 0.15, 0.09] = [5; 3, 2, 1, 1]$, i.e., there can be several *representations* of the same weighted game. We can obtain power indices for weighted games by averaging over all representations of a certain type. If we restrict to integer weights and quota with minimum possible weight sum $\sum_{i=1}^n w_i$, we obtain the *minimum sum representation index* MSRI. We may also average over all normalized weight vectors, i.e., over the polyhedron $P^{\mathrm{w}}(v) = \left\{w \in \mathbb{R}^n_{\ge 0} : \sum_{i=1}^n w_i = 1, w(S) \ge w(T) \,\forall \text{ minimal winning S and all maximal losing T}\right\}$. With this the *average weight index* is given by

$$\mathrm{AWI}(v) = \frac{1}{\int_{P^{\mathrm{w}}(v)} \mathrm{d}\, w} \cdot \left(\int_{P^{\mathrm{w}}(v)} w_1 \,\mathrm{d}\, w, \dots, \int_{P^{\mathrm{w}}(v)} w_n \,\mathrm{d}\, w \right).$$

Taking also the quota into account we can consider $P^{\mathrm{r}}(v) = \left\{(q, w) \in \mathbb{R}^{n+1}_{\ge 0} : \sum_{i=1}^n w_i = 1, q \le 1, w(S) \ge q, \forall \text{ min. win. } S, w(T) \le q \,\forall \text{ max. los. T}\right\}$ and define the *average representation index* as

$$\mathrm{ARI}(v) = \frac{1}{\int_{P^{\mathrm{r}}(v)} \mathrm{d}(q, w)} \cdot \left(\int_{P^{\mathrm{r}}(v)} w_1 \,\mathrm{d}(q, w), \dots, \int_{P^{\mathrm{r}}(v)} w_n \,\mathrm{d}(q, w) \right).$$

All those three representation based power indices are positive, efficient and symmetric. The null voter property is only satisfied for the MSRI.

3. Results

For every positive, efficient power index that satisfies the null player property the power of a dictator is exactly one. In this case, we speak of full power. So, the largest possible power for a player is 1 and it is quite natural to ask for the largest possible power of a player that is strictly less than 1. Since the number of simple games is finite for each number $n \in \mathbb{N}$ of players, the answer is a well-defined number, which possibly depends on n. If v is a simple game with $n \ge 2$ players and player i is not a dictator, then there exists a player $j \ne i$ that is contained in some minimal winning coalition S. Thus, for the Shapley–Shubik, the Penrose–Banzhaf, the Public Good index, the Johnston index, and the Deegan–Packel index every player with power 1 is a dictator. So, the condition that player i is not a dictator is equivalent to $p_i(v) < 1$ in the following four theorems. Moreover, $n \ge 2$ is implied for the number of players. As preparation we observe:

Lemma 1. *If v is a simple game with player set N, $v(N \setminus \{i\}) = 0$, and $v(\{i\}) = 1$, then player i is a dictator.*