Economics Letters 166 (2018) 18–24

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Mean growth and stochastic stability in endogenous growth models[☆]

Raouf Boucekkine [a], Patrick A. Pintus [b,a,*], Benteng Zou [c]

[a] Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE, France
[b] CNRS-InSHS, France
[c] CREA, University of Luxembourg, Luxembourg

ARTICLE INFO

Article history:
Received 8 February 2018
Accepted 9 February 2018
Available online 19 February 2018

JEL classification:
O40
C61
C62

Keywords:
Endogenous stochastic growth
Mean growth
Stochastic stability
AK model
Global diversification

ABSTRACT

Under uncertainty, mean growth of, say, wealth is often defined as the growth rate of average wealth, but it can alternatively be defined as the average growth rate of wealth. We argue that stochastic stability points to the latter notion of mean growth as the theoretically relevant one. Our discussion is cast within the class of continuous-time *AK*-type models subject to geometric Brownian motions. First, stability concepts related to stochastic linear homogeneous differential equations are introduced and applied to the canonical *AK* model. It is readily shown that exponential balanced-growth paths are not robust to uncertainty. In a second application, we evaluate the quantitative implications of adopting the stochastic-stability-related concept of mean growth for the comparative statics of global diversification in the seminal model due to Obstfeld (1994).

1. Introduction

In stochastic growth modelling, the concepts of mean growth and growth volatility are of course central, and there exists a related and vast, empirical and theoretical, literature (Ramey and Ramey, 1995, for example). This paper is concerned with a key conceptual question that, to our knowledge, has not been properly addressed: how should mean growth be defined? As growth volatility is nothing but the measurement of deviations from mean growth, the latter conceptual question is of the utmost importance. To make our arguments mathematically clear, we shall illustrate our arguments using the class of continuous time *AK*-type stochastic models, which feature the benchmark endogenous growth structure, a widely chosen framework in the literature (Obstfeld, 1994; Jones and Manuelli, 2005; Steger, 2005, or Boucekkine et al., 2014). It should be recalled here that the *AK* structure is the reduced form of most endogenous one-sector growth models, ranging from learning-by-doing settings to R&D-based growth models, including those with human capital or public capital accumulation (see Barro and Sala-i Martin, 1995, chapters 4, 6 and 7). Last but not least, because of the knife-edge property of endogenous growth, models that do not have an *AK* reduced form generally converge to this form after transitional dynamics, see for example Jones and Manuelli (1990). Therefore, studying the conditions for stochastic stability in this class of models is relevant.

To formulate accurately our research question, suppose we are concerned with the growth of an economic variable, say wealth, in an *AK*-economy subject to external shocks, typically modelled as geometric Brownian motions in the literature. In such a setting, how should we define mean growth? Is it the growth rate of average (or expected) wealth, as it is generally the case in the economic literature cited just above? Or alternatively the average (or expected) growth rate of wealth? It is important to note that in general there is no degree of freedom behind the questions above, one cannot choose freely between the two definitions. For example if wealth were log-normally distributed, it follows by Jensen's inequality that the average growth rate of wealth – second notion – is lower than the growth rate of average wealth — first notion. In this paper, we show that stochastic stability points to the average (or expected) growth rate of wealth as the theoretically relevant concept for mean growth.

More specifically, we claim that we can safely discriminate between the two definitions using the concept of stochastic stability within the class of models used in endogenous growth theory: *AK*-type models usually deliver linear stochastic differential equations for which a large set of mathematical tools is available. It is

☆ We thank Giorgio Fabbri, Cuong Le Van, Takashi Kamihigashi, Maurice Obstfeld, Fabien Prieur, John Stachurski and Thomas Steger for useful discussions. This paper supersedes a previous version titled "Stochastic Stability of Endogenous Growth: Theory and Applications". The usual disclaimer applies.

* Correspondence to: 5-9 Boulevard Bourdet, CS 50498 13205 Marseille Cedex 1, France.

E-mail addresses: Raouf.Boucekkine@univ-amu.fr (R. Boucekkine), Patrick.Pintus@univ-amu.fr (P.A. Pintus), benteng.zou@uni.lu (B. Zou).

https://doi.org/10.1016/j.econlet.2018.02.014
0165-1765/

worth pointing out at this stage that while neoclassical stochastic growth models have been the subject of a quite visible literature (see Brock and Mirman, 1972; Mirman and Zilcha, 1975, or Merton, 1975), no such a literature exists for endogenous growth models. This is partly due to the fact that many of these models rely on zero aggregate uncertainty as in the early R&D based models (see for example, Barro and Sala-i Martin, 1995, chapters 6 and 7). When uncertainty does not vanish by aggregation as in de Hek (1999), the usual treatment consists in applying Merton's portfolio choice methodology (Merton, 1969, 1971) to track mean growth and growth volatility as in Obstfeld (1994) and Steger (2005) or more recently Boucekkine et al. (2014). Within this methodology, stochastic stability is not an issue, and as in Obstfeld (1994), the analysis of mean growth usually relies on the traditional latter definition (as growth rate of average, or expected, magnitudes).

It is important to stress at this stage that one cannot address the issue of stochastic stability of endogenous growth simply by adapting the available proofs in Brock and Mirman (1972) or Merton (1975). For example, strict concavity of the production function is needed in the latter in order to build up the probability measure for stability in distribution, so the strategy cannot be applied to the benchmark stochastic endogenous growth model, the AK model with random output technology. Rather, we simply rely on the specialized mathematical literature on linear stochastic differential equations (Mao, 2011, or Khasminskii, 2012), and we are able to straightforwardly state stochastic stability theorems. We then start illustrating these theorems using the standard stochastic *AK* model (Steger, 2005). Strikingly enough, we ultimately show that the typical (deterministic) balanced growth paths are hardly stochastically stable in our simple framework. Even more, we show that the trivial equilibrium, $k^* = 0$, is globally stochastically asymptotically stable in the large and almost surely exponentially stable (that is, the optimal paths almost surely collapse at exponential speed) even when productivity is arbitrarily high. Kamihigashi (2006) states a similar convergence result for discrete time stochastic growth models. However, as it transpires from the main result of that paper (Theorem 2.1, page 233, in Kamihigashi, 2006), such a discrete-time setting requires a bunch of nontrivial conditions. Our continuous time framework allows to reach the same conclusion at a definitely much lower analytical cost. That said, Kamihigashi's work is extremely worthwhile in that it shows that the methodological problems outlined in this note are not specific to continuous-time frameworks.

More importantly, we claim that choosing this or that definition of mean growth matters for the economic outcomes generated within a model. To give a second stark example, we revisit Obstfeld (1994)'s model on the virtues of global diversification. Not surprisingly, stochastic stability holds if and only if the average growth rate is positive, a condition that is stronger than the requirement that the growth rate of average wealth be positive. More importantly, we show that very different comparative statics results obtain when one uses our proposed definition of mean growth, as one should in view of stability conditions. More precisely, mean growth happens to be enhanced by financial integration under conditions that would possibly lead to the opposite conclusion if one were to use the definition of mean growth advocated in Obstfeld (1994). This property is most striking in a specialized economy, where for example a fall in exogenous risk results in larger growth even if the intertemporal substitution elasticity is smaller than one, despite the fact that a portfolio shift does not happen.

This paper is organized as follows. Section 2 briefly reviews the main mathematical result and a first application to the stochastic *AK* model. Section 3 presents a second application to a global diversification model. Section 4 concludes. In the Appendix, we present the general mathematical definitions and results.

2. Stability of linear stochastic differential equations with an application to the *AK* model

2.1. Basic mathematical concepts and properties

Consider the typical linear Ito stochastic differential equation

$$dx(t) = ax(t)dt + bx(t)dB(t), \ \ t \geq 0$$

with initial condition $x(0) = x_0$ given, $B(t)$ a standard Brownian Motion, a and b constants. The general solution takes the form

$$x(t) = x_0 \ \exp\left\{\left(a - \frac{b^2}{2}\right)t + bB(t)\right\}. \tag{1}$$

Compared to the pure deterministic case (with $b = 0$), an extra negative term, $-\frac{b^2}{2}$, shows up in the deterministic part of the solution. It is therefore easy to figure out why the noise term, $bx(t)dB(t)$, is indeed stabilizing. Incidentally, introducing some specific white noises is one common way to "stabilize" dynamic systems. The pioneering work is from Khasminskii (2012) and some more recent results can be found in Appleby et al. (2008) and references therein. Thus, the stability conditions under stochastic environments may well differ from the case with certainty. We present the general definitions and results in the Appendix, and here we only show the simplest version which is sufficient for the current study, with constant-coefficient stochastic differential equations.

Proposition 1. *Consider the homogeneous linear stochastic equation,* $dx(t) = ax(t)dt + bx(t)dB(t), \ t \geq 0$*; its equilibrium solution* $x^* = 0$ *is stochastically stable if and only if*

$$a < \frac{b^2}{2}.$$

This result reads that if $a < \frac{b^2}{2}$, then almost all sample paths of the solution will tend to the equilibrium solution $x^* = 0$ and such a convergence is exponentially fast, which is obviously not the case in the deterministic case if $a > 0$. This is the key point behind the striking results on the stochastic stability of balanced growth paths in the *AK* model shown here below.

2.2. Stochastic stability in the AK growth model

Consider a strictly increasing and strictly concave utility U and

$$\max_c E_0 \int_0^\infty U(c)e^{-\rho t}dt, \tag{2}$$

subject to

$$dk(t) = (Ak(t) - c(t) - \delta k(t))dt + bAk(t)dW(t), \quad \forall t \geq 0 \tag{3}$$

with given initial condition $k(0) = k_0$, positive constants δ and ρ that measure depreciation and time preference, b is a parameter governing volatility and $W(t)$ is one-dimensional Brownian motion. Define Bellman's value-function as

$$V(k, t) = \max_c E_t \int_t^\infty u(c)e^{-\rho t}dt.$$

Then this value function must satisfy the following stochastic Hamilton–Jacobi–Bellman equation

$$\rho V(k) = \max_c \left\{U(c) + V_k \cdot (Ak - c - \delta k) + \frac{1}{2}b^2A^2k^2V_{kk}\right\} \tag{4}$$

with V_k the first order derivative with respect to k. First order condition on the right hand side of (4) yields

$$U'(c) = V_k(k). \tag{5}$$