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Asymmetric equilibria in spatial competition under weakly concave scoring rules

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HIGHLIGHTS

- We investigate Hotelling–Downs spatial competition under scoring rules.
- Non-convergent equilibria can exist for the class of weakly concave scoring rules.
- However, no symmetric equilibria exist-any equilibrium must be asymmetric.
- Over half the agents locate at one of the extreme-most occupied positions.

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ABSTRACT

The Hotelling–Downs model of spatial competition is used to investigate the strategic position-taking behavior of firms or political parties under scoring rules. Previous studies of non-convergent Nash equilibria – equilibria in which divergent positions are chosen – found that they often do not exist and, when they do, they are fairly symmetric. In particular, this is true for convex scoring rules (Cahan and Slinko, 2017). Here, we investigate non-convergent equilibria for the broad class of weakly concave scoring rules. Surprisingly, we find that only asymmetric equilibria can exist, and we present several examples.

1. Introduction

The classical Hotelling–Downs model of competition between firms (political parties) for customers (voters) assumes that customers always buy from the nearest firm (the electoral system is plurality). This situation has been well-studied (Eaton and Lipsey, 1975; Denzau et al., 1985) — equilibria are numerous but certain properties restrict their predictive capabilities, namely, in equilibrium no more than two firms can co-locate. Hence this model cannot fully explain the Principle of Local Clustering (Eaton and Lipsey, 1975). In reality, in such clusters – think about clusters of fast food outlets (candidates grouping into party umbrellas) – there does not seem to be a restriction on the number of co-locating agents. Second, the equilibria in the classical model are quite symmetric, but actual business districts (parties) often differ in size and not all neighborhoods (ideologies) may be equally served.

Cox (1987) and Myerson (1999) investigate more general *scoring rules* in the voting context. Under a scoring rule, each voter ranks all m agents. The agent ranked ith on the voter's ranking receives s_i points. A scoring rule is thus a non-negative m-vector, $s = (s_1, \dots, s_m)$, with $s_1 \geq \cdots \geq s_m \geq 0$ and $s_1 > s_m$. Well-known examples include plurality, Borda and antiplurality, given by $s = (1, 0, \dots, 0)$, $s = (m-1, m-2, \dots, 0)$ and $s = (1, \dots, 1, 0)$, respectively. A scoring rule may alternatively be interpreted as follows (Cox, 1990; Cahan and Slinko, 2017, 2018): s_i is the probability that a consumer (voter) patronizes (votes for) the ith nearest firm (party).[1]

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[1] The results are invariant to scaling the score vector by a positive constant, so we can normalize the score vector, dividing by $\sum_{i=1}^{m} s_i$.

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Scoring rules allow for a more realistic range of consumer behavior, and in some cases large clusters of firms exist in equilibrium. Cox (1990) first characterized convergent Nash equilibria (CNE), in which all agents adopt the same position. Nonconvergent equilibria (NCNE) have been characterized for plurality (Eaton and Lipsey, 1975; Denzau et al., 1985) and for antiplurality, where none exist (Cox, 1987, p. 93). Cahan and Slinko (2017) undertook a more general investigation of NCNE, ruling out NCNE in a few cases, including when the score vector is convex—the only possible exception is the truncated Borda rule. They also find a class of rules allowing NCNE with multiple agents clustered at distinct locations along the interval. Cahan and Slinko (2018) find that the class of best-worst rules have similar NCNE as plurality which, in particular, means plenty of them. Moreover, their NCNE exhibit greater moderation than under plurality.

Here we investigate the broad class of weakly concave scoring rules. A scoring rule is *weakly concave* if it obeys the following property:

$$s_i - s_{i+1} \leq s_{m-i} - s_{m-i+1}, \tag{1}$$

for all $1 \leq i \leq \lfloor m/2 \rfloor$. That is, for every drop between consecutive scores at the top of the ranking, the corresponding drop at the bottom is at least as large. This represents a broad class of plausible probabilistic descriptions of consumer behavior where consumers make larger distinctions at the lower end of their rankings than at the top end. Weakly concave rules encompass the important subclass of concave rules, where $s_1 - s_2 \leq s_2 - s_3 \leq \cdots \leq s_{m-1} - s_m$. That is, high ranked firms are similarly patronized, but consumers become increasingly less likely to patronize firms as we move down the ranking. Such a rule is "worst-punishing" (Cox, 1987) in the sense that it is more important to avoid low rankings than distinguish oneself at the top.

We find that NCNE, if they exist, are remarkably strange, with over half the agents clustering at one of the extreme occupied locations. We indeed find that such asymmetric equilibria exist, and we provide several examples. Asymmetric multiagent equilibrium clusters are quite unusual in the literature, which often finds (or restricts attention to) symmetric equilibria only. Only a few papers address the issue in other contexts. Chisik and Lemke (2006) show that when office motivated candidates who only care about winning are assumed, rather than vote maximizing candidates as assumed here, equilibrium properties under plurality are quite different and, in particular, asymmetric equilibria exist. Xefteris (2016) considers asymmetric equilibria in a model with endogenous candidates under k-vote procedures, where a voter has the option to assign one vote to at most k candidates. He also assumes office motivated candidates and, moreover, k-vote procedures cannot be described as scoring rules.[2]

The model features a unit mass of consumers distributed uniformly on the interval $[0, 1]$ and $m \geq 4$ competing firms. Firm i's selected location is x_i and the vector of all firms' locations is a strategy profile $x = (x_1, \ldots, x_m) \in [0, 1]^m$. Firms move simultaneously and maximize the number of points received from all voters, denoted $v_i(x)$.[3] Let $x^1, \ldots, x^q$ be the distinct locations in x, assuming $x^1 < \cdots < x^q$. In an NCNE, $q \geq 2$. Denote by $n_i \in \mathbb{N}$ the number of firms at x^i. The length of an interval $I = [a, b] \subseteq [0, 1]$ is $\ell(I)$. Consumers have single-peaked, symmetric utility functions, and thus rank firms by distance. Consumers indifferent between firms randomize to choose a strict ranking.

We study pure strategy Nash equilibria: $x^* = (x_1^*, \ldots, x_m^*)$ is an equilibrium if and only if $v_i(x^*) \geq v_i(x_i, x_{-i}^*)$, for all $x_i \in [0, 1]$, where $(x_i, x_{-i}^*) = (x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_m^*)$. We will also denote by x^{i-} and x^{i+} the limits approaching the point x^i from the left and the right, respectively.

2. Results

First, we need two technical lemmas.

Lemma 1. *If s is a weakly concave rule, then*

$$s_j + s_{m-j+1} \geq \frac{1}{j}\left(\sum_{i=1}^{j} s_i + \sum_{i=m-j+1}^{m} s_i\right) \tag{2}$$

for all $1 \leq j \leq \lfloor m/2 \rfloor + 1$. Moreover, if s satisfies

$$s_k + s_{m-k+1} \geq \frac{1}{k}\left(\sum_{i=1}^{k} s_i + \sum_{i=m-k+1}^{m} s_i\right) \tag{3}$$

for some $k > \lfloor m/2 \rfloor + 1$, then (2) holds for all j with $1 \leq j \leq k$.

Proof. Let $1 \leq j \leq \lfloor m/2 \rfloor + 1$. Eq. (1) implies

$$s_1 + s_m \leq s_2 + s_{m-1} \leq \cdots \leq s_j + s_{m-j+1},$$

whence

$$\sum_{i=1}^{j} s_i + \sum_{i=m-j+1}^{m} s_i = \sum_{i=1}^{j}(s_i + s_{m-i+1}) \leq j(s_j + s_{m-j+1}),$$

which, dividing by j, gives (2).

Suppose (3) holds for some $k > \lfloor m/2 \rfloor + 1$. If we prove that (2) holds for $j = k-1$, the result will follow by induction. If $j = k-1 = \lfloor m/2 \rfloor + 1$, it follows from the first part of the lemma. So assume $k > \lfloor m/2 \rfloor + 2$. Then

$$\left(\sum_{i=1}^{k} s_i + \sum_{i=m-k+1}^{m} s_i\right) - \left(\sum_{i=1}^{k-1} s_i + \sum_{i=m-k+2}^{m} s_i\right)$$
$$= s_k + s_{m-k+1} \geq \frac{1}{k}\left(\sum_{i=1}^{k} s_i + \sum_{i=m-k+1}^{m} s_i\right).$$

Rearranging,

$$\frac{1}{k}\left(\sum_{i=1}^{k} s_i + \sum_{i=m-k+1}^{m} s_i\right) \geq \frac{1}{k-1}\left(\sum_{i=1}^{k-1} s_i + \sum_{i=m-k+2}^{m} s_i\right). \tag{4}$$

Since $k > \lfloor m/2 \rfloor + 2$, we have $m - k + 1 \leq \lfloor m/2 \rfloor$ and hence by (1) we conclude $s_{m-k+1} + s_k \leq s_{m-k+2} + s_{k-1}$. Combining this, the assumption that (3) holds for k, and (4), we have

$$s_{m-k+2} + s_{k-1} \geq \frac{1}{k-1}\left(\sum_{i=1}^{k-1} s_i + \sum_{i=m-k+2}^{m} s_i\right).$$

Thus, (2) holds for $j = k-1$. □

Lemma 2. *(a) If $n_1 = 2$ or $n_q = 2$, a necessary condition for* NCNE *is $s_2 = s_{m-1}$. (b) Given scoring rule s, let $1 \leq k \leq m-1$ be such that $s_1 = \cdots = s_k > s_{k+1}$. Then a necessary condition for* NCNE *is* $\min(n_1, n_q) > k$.

Proof. For (a), if $n_1 = 2$ then the unit interval can be partitioned into subintervals I_t of consumers that rank firm 1 the same, with firm 1 obtaining $(s_l + s_{l+1})\ell(I_t)/2$ from consumers in I_t for some l. By deviating infinitesimally left or right of x^1, firm 1 will now

[2] Informally, we could imagine a k-vote procedure as the scoring rule $s = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, except that individual voters need not assign all k votes.

[3] Since the total number of points is fixed at $\sum_{i=1}^{m} s_i$, $v_i(x)$ is (with the appropriate normalization) candidate i's share of the total number of points up for grabs or, in the probabilistic interpretation, the expected number of patrons.