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On bargaining sets of supplier-firm-buyer games

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HIGHLIGHTS

- We study a special three-sided matching game, the so-called supplier-firm-buyer game.
- We show that on this class the core and the Davis–Maschler bargaining set coincide.
- Moreover, the core also coincides with the Mas-Colell bargaining set on these games.
- Our results rest on the closedness of this class for taking the maximal excess game.

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ABSTRACT

We study a special three-sided matching game, the so-called supplier-firm-buyer game, in which buyers and sellers (suppliers) trade indirectly through middlemen (firms). Stuart (1997) showed that all supplier-firm-buyer games have non-empty core. We show that for these games the core coincides with the classical bargaining set (Davis and Maschler, 1967), and also with the Mas-Colell bargaining set (Mas-Colell, 1989).

1. Introduction

In their seminal paper Shapley and Shubik (1972) introduced assignment games to study two-sided matching markets where there are indivisible goods which are traded between sellers and buyers in exchange for money. Their proof of the non-emptiness of the core established a fruitful research area. Multi-sided assignment games, however, have different features. Most importantly, the non-emptiness of the core is not guaranteed anymore even when there are only three sides in the game, as first demonstrated by Kaneko and Wooders (1982).

Since the core may be empty for multi-sided assignment games, some authors study conditions to obtain the non-emptiness of the core (see for instance Quint, 1991; Stuart, 1997; Sherstyuk, 1999; Atay and Núñez, 2017). In this paper, we focus on the class introduced by Brandenburger and Stuart (1996) and investigated by Stuart (1997). In the so-called supplier-firm-buyer games, agents in the market are partitioned into three sides and the groups are arranged along a chain. Sellers (suppliers) and buyers (customers) are at the two ends of the chain, but trade between them can only be made via agents in the middle (firms). The valuation on the supplier-firm-buyer triplets is locally additive, it sums up the potential values of the supplier-firm and of the firm-buyer matchings, but it is realized only if all three parties cooperate. Stuart (1997) showed that all supplier-firm-buyer games have non-empty core.

In order to find plausible payoff allocations even in games with empty core, Aumann and Maschler (1964) suggested a set-valued solution concept that incorporates some negotiating possibilities of the players. Among the various bargaining sets proposed, the one investigated by Davis and Maschler (1967) has emerged, for it was proved to be non-empty whenever the game has a non-empty imputation set (Davis and Maschler, 1967). Mas-Colell (1989) introduced another bargaining set notion based on preimputations

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and showed that it is non-empty for any game. (Holzman, 2001) proved that for superadditive games the classical (Davis–Maschler) bargaining set is included in the Mas-Colell bargaining set.

Solymosi (1999) presented a necessary and sufficient condition in terms of the so-called maximal excess games for the coincidence of the classical bargaining set and the core in superadditive games. Applied for two-sided assignment games, Solymosi (1999) proved the coincidence of the classical bargaining set and the core, by using the result of Granot and Granot (1992) who showed that the class of two-sided assignment games is closed for taking the maximal excess game at any imputation. Solymosi (2008) extended this closedness result to all preimputations in classes of partitioning games defined on a fixed family of basic coalitions and, by using the characterization by Holzman (2001) of the coincidence between the Mas-Colell bargaining set and the core, established even this stronger equivalence result for certain subclasses of partitioning games, including the two-sided assignment games.

In this paper, following a similar approach, we show that the class of supplier-firm-buyer games is closed for taking (the 0-normalization of) the maximal excess game at any (pre)imputation. Then, we establish the coincidence between the classical bargaining set and the core, and moreover the coincidence between the Mas-Colell bargaining set and the core for supplier-firm-buyer games. We restrict ourselves to the supplier-firm-buyer case, but all the arguments and results in the paper can be extended to m-sided assignment games with locally additive evaluation on the basic path-coalitions consisting exactly one agent from each side of the market. In real life, we observe markets that consist of more than two sides where the sectors are organized in a line. In such markets, agents from the same industry have an industry specific role and hence we cannot study these markets as separated two-sided markets. Thus, we believe that the generalization of supplier-firm-buyer games, namely multi-sided assignment games with locally additive value functions, is a useful model to study value generation and allocation in supply chains.

2. Preliminaries

A *transferable utility cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *0-normalized* if $v(\{i\}) = 0$ for every $i \in N$. It is *superadditive* if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$ for every two coalitions $S, T \subseteq N$.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i\in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$; *individually rational*, if $x_i = x(\{i\}) \geq v(\{i\})$ for all $i \in N$; *coalitionally rational*, if $x(S) \geq v(S)$ for all $S \subseteq N$. The set of *preimputations*, $I^*(v)$, consists of the efficient payoff vectors, the set of *imputations*, $I(v)$, consists of the individually rational preimputations, and the *core*, $C(v)$, is the set of coalitionally rational (pre)imputations. We call a game *balanced* if its core is non-empty.

Given a game (N, v), the *excess* of a coalition $S \subseteq N$ at a payoff allocation x is $e_x(S) = v(S) - x(S)$. It is a measure of gain (or loss) to S, if its members disagree on x and leave it to form their own coalition. On player set N, games v and w are *strategically equivalent*, if there exist $\alpha > 0$ and $b \in \mathbb{R}^N$ such that $w(S) = \alpha v(S) + \sum_{i\in S} b_i$ for all $S \subseteq N$. In particular, the 0-normalization of v, denoted by v^0, is obtained when $\alpha = 1$ and $b = (b_i = -v(\{i\}) : i \in N)$. Clearly, v is balanced if and only if v^0 is balanced.

Aumann and Maschler (1964) argued that the purpose of the game is to reach some kind of stability, to which the players would or should agree, if they want any agreement. This stability should reflect in some sense the power of each player, but should be weaker than the sometimes too strong stability the core outcomes capture. Aumann and Maschler (1964) considered several bargaining sets as reasonable outcomes of negotiations among coalitions versus coalitions. Davis and Maschler (1967) investigated another variant, denoted $\mathbf{M}_1^i$, where individuals bargain with individuals and proved its non-emptiness under the very mild condition that the game has imputations. Hence, it received most attention and became the classical bargaining set. The idea behind is that an allocation can be considered stable (even if not in the core) if all objections raised by some player can be nullified by another player.

Let (N, v) be a coalitional game, $x \in I(v)$ be an imputation, and $i, j \in N$ be two different players. A pair (S, y) where $S \subseteq N$ and $y \in \mathbb{R}^S$ is an *objection of i against j at x* if $i \in S, j \notin S, y(S) = v(S)$, and $y_l > x_l$ for all $l \in S$. Then, a *counter-objection of j to the objection (S, y) of i at x* is a pair (T, z) such that $T \subseteq N$ and $z \in \mathbb{R}^T$ where $j \in T, i \notin T, z(T) = v(T), z_k \geq y_k$ for all $k \in T \cap S$, and $z_l \geq x_l$ for all $l \in T \setminus S$. An objection is *justified* (in the Davis–Maschler sense) if there does not exist any counter-objection to it. With these notions of objection and counter-objection, Davis and Maschler (1967) introduced what is known as the *classical bargaining set* $\mathbf{M}_1^i$.

Definition 1 (*Davis and Maschler, 1967*)**.** Let (N, v) be a coalitional game. The *classical bargaining set* is the set of imputations at which there is no justified objection:

$$\mathbf{M}_1^i(v) = \{x \in I(v) \mid \text{ for every objection at } x \text{ there is a counter-objection}\}.$$

Since no objections, hence no justified objections can be raised at core imputations, the core is always a subset of the classical bargaining set. Maschler (1976) discussed a five-player market game for which the bargaining set is a strict superset of the core, moreover, "for which the bargaining set seems to be intuitively more acceptable than the (non-empty) core". On the other hand, Solymosi (2002) proved that in (at most) 4-player games, if the core is non-empty, it coincides with the classical bargaining set.

Another bargaining set notion was introduced by Mas-Colell (1989). In that concept coalitions bargain, rather than pairs of players. Moreover, all efficient payoff vectors are considered, the individual rationality requirement is dropped. Thus, the notions of objection and counter-objection are modified.

Let (N, v) be a coalitional game. Given a preimputation $x \in I^*(N, v)$, we say that a pair (S, y) where $\emptyset \neq S \subseteq N$ and $y \in \mathbb{R}^S$ is a *weak objection* if $y(S) = v(S)$ and $y_l \geq x_l$ for all $l \in S$ with at least one strict inequality for some $l \in S$. Then, a pair (T, z) where $\emptyset \neq T \subseteq N$ and $z \in \mathbb{R}^T$ is a *strong counter-objection* to objection (S, y) at x if $z(T) = v(T)$ and $z_l \geq y_l$ for all $l \in T \cap S$, $z_l \geq x_l$ for all $l \in T \setminus S$ with at least one strict inequality for some $l \in T$. Using these concepts of weak objection and strong counter-objection, Mas-Colell (1989) introduced a notion of bargaining set.

Definition 2 (*Mas-Colell, 1989*)**.** Let (N, v) be a coalitional game. The *Mas-Colell bargaining set* is the set of preimputations such that every weak objection at the given preimputation can be strongly countered:

$$\mathbf{M}_{MC}^* = \{x \in I^*(v) | \text{every weak objection at } x \text{ can be strongly countered}\}.$$

Mas-Colell (1989) showed that the Mas-Colell bargaining set is non-empty in any game, and a superset of the core. Mas-Colell (1989) presented a 4-player market game where the Mas-Colell bargaining set contains imputations outside the (non-empty) core. On the other hand, it is easily seen that in (at most) 3-player games,