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# Calculating direct and indirect contributions of players in cooperative games via the multi-linear extension

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## HIGHLIGHTS

- We consider solutions for cooperative games with transferable utility.
- Casajus and Huettner (2017, GEB) introduce the resolution of such solutions.
- The members of a resolution can be expressed in terms of the multilinear extension.
- Their potentials also can be expressed in terms of the multilinear extension.

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## ABSTRACT

The resolution of a solution for cooperative games is a recently developed tool to decompose a solution into a player's direct contribution in a game and her (higher-order) indirect contribution, i.e., her contribution to other players' direct contributions. We provide new formulae for resolutions and their potentials, which facilitate the calculation of them in large (voting) games. These formulae make use of the multi-linear extension of cooperative games with transferable utility.



## 1. Introduction

The Shapley value (Shapley, 1953) is probably the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). Hart and Mas-Colell (1989) suggest an indirect characterization as marginal contributions of a potential (function).[1] Roth (1977) shows that the Shapley value can be understood as a von Neumann–Morgenstern utility. As a contribution to the Nash program, which aims at building bridges between cooperative and non-cooperative game theory, Pérez-Castrillo and Wettstein (2001) implement the Shapley value as the outcomes of the sub-game perfect equilibria of a combined bidding and proposing mechanism, which is modelled by a non-cooperative extensive form game.[2]

Recently, Casajus and Huettner (forthcoming) suggest the decomposition of solutions as a tool to rely the Shapley value to the

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[1] Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.

[2] Ju and Wettstein (2009) suggest a class of bidding mechanisms that implement several solution concepts for TU games.

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"naïve solution" that assigns to each player her marginal contribution to the coalition of *all others*. A solution $\psi$ decomposes a solution $\varphi$ if it splits $\varphi$ into direct and indirect contributions in the following sense. A particular player's payoff for $\varphi$ is the sum of her payoff for $\psi$ (direct contribution) and what the other players gain or lose under $\psi$ when this particular player leaves the game (indirect contribution). That is, the indirect contribution reflects what a player contributes to the other players' direct contributions. The property of a solution to be decomposable, i.e., to admit a decomposer is equivalent to a number of other well-known properties of solutions (Casajus and Huettner, forthcoming, Theorem 4): balanced contributions (Myerson, 1980), path independence (Hart and Mas-Colell, 1989), consistency with the Shapley value (Calvo and Santos, 1997), and admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998). It turns out that the Shapley value is the unique decomposable decomposer of the naïve solution (Casajus and Huettner, forthcoming, Theorem 3). More generally, any decomposable solution $\varphi$ admits a unique decomposable decomposer (Casajus and Huettner, forthcoming, Proposition 2) and therefore a unique resolution, i.e., a sequence of solutions starting with $\varphi$ itself and such that any member of this sequence is decomposed by its successor (Casajus and Huettner, forthcoming, Theorem 7(i)).

Owen (1972) introduces the multi-linear extension of a TU game. The domain of this extension is the standard cube, representing the players' probabilities of participating in the generation of worth. He obtains the Shapley value as the integral of partial derivatives of the multi-linear extension alongside the diagonal of the standard cube. This formula is particularly useful when computing the Shapley value for large (voting) games (Owen, 1972; Leech, 2003). Casajus and Huettner (2015) show that the potential of the Shapley value can be expressed as the integral of the total derivative of the multi-linear extension alongside the diagonal of the standard cube.

In this paper, we generalize the above mentioned results of Owen (1972) and Casajus and Huettner (2015). We express the members of the resolution of the Shapley value and of any other decomposable solution in terms of the partial derivatives of the multi-linear extension alongside the diagonal of the standard cube. Moreover, their zero-normalized potentials can also be expressed in terms of the total differential.

This note is organized as follows. Basic definitions and notation are given in the second section. In the third section, we provide the definitions and results on the decomposition and resolutions of solutions. The fourth section contains our new results. Some remarks conclude this note. The proof of our main result is contained in the Appendix.

## 2. Basic definitions and notation

A (TU) game on a finite player set $N$ is given by a *characteristic function* $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. The set of all games on $N$ is denoted by $\mathbb{V}(N)$. Let $\mathcal{N}$ denote the set of all finite player sets.[3] The cardinalities of $S, T, N, M \in \mathcal{N}$ are denoted by $s$, $t$, $n$, and $m$, respectively.

For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}(N)$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a *unanimity game*. As pointed out in Shapley (1953), these unanimity games form a basis of the vector space[4] $\mathbb{V}(N)$, i.e., any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T., \tag{1}$$

where the *Harsanyi dividends* $\lambda_T(v)$ can be determined recursively via $\lambda_T(v) = v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v)$ for all $T \subseteq N$, $T \neq \emptyset$ (see Harsanyi, 1959).

A *solution/value* is an operator $\varphi$ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. The *Shapley value* (Shapley, 1953) distributes the dividends $\lambda_T(v)$ equally among the players in $T$, i.e.,

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t} \tag{2}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. A solution is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$.

In this paper, we consider situations where a player leaves the game. For $v \in \mathbb{V}(N)$ and $i \in N$, the *restriction* of $v$ to $N \setminus \{i\}$ is denoted by $v^{-i} \in \mathbb{V}(N \setminus \{i\})$ and is given by $v^{-i} = v(S)$ for all $S \subseteq N \setminus \{i\}$.

## 3. Decomposition, resolution, and potential

In order to identify the direct and the indirect contributions of players with respect to a given solution, Casajus and Huettner (forthcoming) introduce the notion of a decomposer of a solution.

**Definition 1.** A solution $\psi$ is a *decomposer* of the solution $\varphi$ if

$$\varphi_i(v) = \psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right] \tag{3}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. A solution $\varphi$ is called *decomposable* if there exists a decomposer $\psi$ of $\varphi$.

In this definition, the expression $\psi_i(v)$ reflects player $i$'s direct contribution subsumed under the solution $\varphi$, while the expression $\sum_{\ell \in N \setminus \{i\}} \left[\psi_\ell(v) - \psi_\ell\left(v^{-i}\right)\right]$ reflects player $i$'s indirect contributions, i.e., her contribution to the direct contributions of the other players. Casajus and Huettner (forthcoming) consider resolutions of solutions in order to study higher-order indirect contributions, e.g., player $i$'s contributions to player $j$'s contributions to player $k$.

**Definition 2.** A *resolution* of a solution $\varphi$ is a sequence $\left(\varphi^{(k)}\right)_{k \in \mathbb{N}}$ of solutions such that $\varphi^{(0)} = \varphi$ and $\varphi^{(k+1)}$ is a decomposer of $\varphi^{(k)}$ for all $k \in \mathbb{N}$. If a resolution exists for a solution, then the latter is called *resolvable*.

It turns out that decomposability and resolvability are equivalent and that a resolution, if it exists, is unique (Casajus and Huettner, forthcoming, Proposition 5 and Theorem 7(i)). Therefore, the solution $\varphi^{(k)}$ is called the *kth decomposer of* $\varphi$. Casajus and Huettner (forthcoming, Theorem 7(ii)–(iv)) show that the resolution of any decomposable solution $\varphi$ is given as follows. For all $k \in \mathbb{N}$, $N \in \mathcal{N}$, and $v \in \mathbb{V}(N)$, we have

$$\varphi^{(k)}(v) = \mathrm{Sh}^{(k)}\left(v^{\varphi}\right), \tag{4}$$

where the resolution of the Shapley value is given by

$$\mathrm{Sh}_i^{(k)}(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t^{k+1}} \qquad \text{for all } i \in N \tag{5}$$

and $v^{\varphi} \in \mathbb{V}(N)$ is defined by

$$v^{\varphi}(S) = \sum_{\ell \in S} \varphi_\ell\left(v|_S\right) \qquad \text{for all } S \subseteq N. \tag{6}$$

By Casajus and Huettner (forthcoming, Theorem 4), decomposability is equivalent to the *admittance of a potential* (Calvo and Santos, 1997; Ortmann, 1998): There exists a mapping (the *potential*) $P : \cup_{N \in \mathcal{N}} \mathbb{V}(N) \to \mathbb{R}$ such that $\varphi_i(v) = P(v) - P\left(v^{-i}\right)$ for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. If a solution $\varphi$ admits a

[3] We assume that the player sets are subsets of some given countably infinite set $\mathfrak{U}$, the universe of players; $\mathcal{N}$ denotes the set of all finite subsets of $\mathfrak{U}$.

[4] For $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.