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A note on ex-ante stable lotteries

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HIGHLIGHTS

- Ex-ante stability as a fairness condition is very demanding.
- Precise bounds on the size of the support of ex-ante stable lotteries are given.
- The result can be interpreted as an impossibility result.

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ABSTRACT

We study ex-ante priority respecting (ex-ante stable) lotteries in the context of object allocation under thick priorities. We show that ex-ante stability as a fairness condition is very demanding: Only few agent–object pairs have a positive probability of being matched in an ex-ante stable assignment. We interpret our result as an impossibility result. With ex-ante stability, one cannot go much beyond randomly breaking ties and implementing a (deterministically) stable matching with respect to the broken ties.

1. Introduction

A classical matching problem with many real-world applications is the assignment of indivisible objects to agents where objects are rationed according to priorities. In applications, such as the school choice problem (Abdulkadiroglu and Sönmez, 2003), priorities are often thick, i.e. many agents have the same priority to obtain a certain object. Thus, it can be the case that agents obtain different assignments ex-post, even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This has motivated researchers to study the problem of designing priority respecting lotteries for allocating objects.

A minimal ex-ante fairness requirement for random assignments under priorities is that the lottery should respect the priorities. One way of formalizing this requirement is the following: An agent i has *ex-ante justified envy* if there is an object s where a lower priority agent j has a positive probability of receiving the object and i would rather have the object s than another object which he receives with positive probability in the lottery under consideration. In this case, it would be natural to eliminate the justified envy, i.e. changing the probability shares such that i has a higher chance of receiving s at the expense of the lower ranked agent j. *Ex-ante stability* requires that there is no ex-ante justified envy. In the school choice set-up, ex-ante stability has been introduced by Kesten and Ünver (2015). For the classical marriage model the condition was first considered by Roth et al. (1993). He et al. (forthcoming) define an appealing class of mechanisms that implement ex-ante stable lotteries.

Even though ex-ante stability is, in a sense, a minimal ex-ante fairness requirement, it is demanding. In an environment with strict priorities (no ties) and where each school has one seat to allocate, Roth et al., 1993 (Corollary 21) prove that each student has a positive probability of receiving a seat at, at most, two schools. In other words, an ex-ante stable lottery is almost deterministic. We generalize this result to the more general set-up with quotas and ties. With strict priorities, we show that an ex-ante stable lottery is almost degenerate, since

- each agent has a positive probability at at most *two* distinct objects for receiving a copy of that object.

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- For each object all but possibly *one* copy are assigned deterministically. For the one copy that is assigned by a lottery, two agents have a positive probability of receiving it.

With ties in the priorities, ex-ante stability is naturally less demanding. However, ex-ante stability imposes a lot of structure on the lottery. We show that the size of the support of an ex-ante stable lottery (the number of pairs being matched with positive probability) is determined by the number of ties the lottery "uses" (i.e. how many agents who have equal priority at some object are matched with positive probability to that object). More precisely, we show that for each ex-ante stable lottery the size of the support is determined by the size of the "cut-off" priority classes: Here, cut-off priority classes are the lowest priority classes at an object, such that an agent of that priority class gets that object with positive probability.

The proofs in this paper use the graph representation of assignment problems due to Balinski and Ratier (1997). As far as we know, this representation has not been used so far in the study of lotteries. We think that our results demonstrate the usefulness of this particular representation for the study of random assignments with priorities.

2. Model

There is a set of n agents N and a set of m object types M. A generic agent is denoted by i and a generic object type by j. Of each object type j, there is a finite number of copies $q_j \in \mathbb{N}$. We assume that there are as many objects as agents, $\sum_{j\in M} q_j = n$.[1] Each agent i has strict preferences P_i over different types of objects. Each object type j has a strict priority ranking $\succ_j$ of agents. Later in Section 3.2 we will also consider the case where object types have indifferences in their priorities.

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a probability distribution over deterministic assignments. By the Birkhoff–von Neumann Theorem, each random assignment corresponds to a bi-stochastic matrix and, vice versa, each such matrix corresponds to a random assignment (see Kojima and Manea, 2010) for a proof in the set-up that we consider). Thus each random assignment is represented by a matrix $\Pi = (\pi_{ij}) \in \mathbb{R}^{N\times M}$ such that

$$0 \le \pi_{ij} \le 1, \quad \sum_{j\in M} \pi_{ij} = 1, \quad \sum_{i\in N} \pi_{ij} = q_j,$$

where π_{ij} is the probability that agent i is matched to an object of type j. The **support** of Π is the set of all non-zero entries of the matrix Π, i.e.

$$\text{supp}(\Pi) := \{ij \in N \times M : \pi_{ij} \neq 0\}.$$

We say that agent i is **fractionally matched** to object type j if there is a positive probability of the pair being matched but they are not matched for sure, i.e. $0 < \pi_{ij} < 1$. A random assignment represented by the matrix $\Pi = (\pi_{ij})$ is **ex-ante blocked** by agent i and object type j if there is some agent $i' \neq i$ with $\pi_{i'j} > 0$ and $i \succ_j i'$ and some object type j' with $\pi_{ij'} > 0$ and $j P_i j'$. A random assignment is **ex-ante stable** if it is not blocked by any agent–object type pair.[2]

[1] Our results can be generalized to the case where the number of objects and agents differ by adding dummy agents and objects. See Aziz and Klaus (2017), for the details of this construction.

[2] For deterministic assignments, ex-ante stability is equivalent to the usual notion of a stable matching. In particular, ex-ante stable assignments always exists, since stable matchings always exist.

2.1. Graph representation

Next, we introduce the graph representation of Balinski and Ratier (1997). In the following, a directed graph Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite set of vertices and $E(\Gamma)$ is a set of ordered pairs of vertices called arcs. For a random assignment Π, we construct a directed graph $\Gamma(\Pi)$ as follows: The vertices are the agent–object type pairs,

$$V = N \times M.$$

There are two kind of arcs. A **horizontal arc** connects two vertices ij and ij' that contain the same agent. A **vertical arc** connects two vertices ij and $i'j$ that contain the same object type. The direction of the arc is determined by the preferences respectively priorities. A horizontal arc points to the more preferred object type according to the agent's preferences. A vertical arc points to the agent with higher priority in the object type's priority. Moreover we only consider those arcs which origin in a pair ij with $\pi_{ij} > 0$. Thus

$$E(\Pi) := \{(ij, i'j') \in V^2 : \pi_{ij} > 0, (i = i', j' P_i j \text{ or } j = j', i' \succ_j i)\}.$$

Immediately from the definition of ex-ante stability we obtain the following necessary and sufficient condition for ex-ante stability (see Fig. 1).

Lemma 1. *If $\Pi = (\pi_{ij})$ is ex-ante stable, then there cannot exist both a horizontal arc (ij', ij) and a vertical arc $(i'j, ij)$ in $\Gamma(\Pi)$ pointing to ij.*

3. Results

3.1. Strict priorities

We are ready to state and prove the main results for the case with strict priorities. First we show that if Π represents an ex-ante stable random assignment, then it has small support.

Proposition 1. *If priorities are strict, then for each ex-ante stable random assignment Π we have*

$$|supp(\Pi)| \le n + m.$$

Proof. We prove the proposition by a double counting argument. Let $U \subseteq V$ be the set of vertices ij that have an incoming horizontal arc in $\Gamma(\Pi)$ and positive probability $\pi_{ij} > 0$. For each $i \in N$, let $M_i(\Pi) \subseteq M$ be the set of object types j such that ij has an incoming horizontal arc and $\pi_{ij} > 0$. By definition, we have $|U| = \sum_{i\in N} |M_i(\Pi)|$. Let $i \in N$. Either i is deterministically matched or he is fractionally matched to multiple object types. In the first case, we have $M_i(\Pi) = \emptyset$. In the second case, let $j \in M_i(\Pi)$ be the least preferred object type (according to i's preferences) among the object types that are fractionally matched to i under Π. Since j is i's least preferred object type to which he is matched, there is for each such object type $j' \neq j$ a horizontal arc pointing from ij to ij'. Thus, in either case, $|\text{supp}(\Pi_i)| - 1 = |M_i(\Pi)|$ where $\text{supp}(\Pi_i)$ is the support of the i-row of Π. Summing over N we obtain

$$\text{supp}(\Pi) - n \le \sum_{i\in N} |M_i(\Pi)| = |U|. \tag{1}$$

Next we bound $|U|$ from above. Let $j \in M$ and $i, i' \in N$. Suppose $\pi_{ij} > 0$, $\pi_{i'j} > 0$ and furthermore that there is a horizontal arc pointing to ij and another horizontal arc pointing to $i'j$. If there were a vertical arc pointing from ij to $i'j$, we would have a contradiction to Lemma 1 and vice versa if there were a vertical arc pointing from $i'j$ to ij, we would also have a contradiction to Lemma 1. Thus for each j there is at most one agent i such that $\pi_{ij} > 0$ and ij has an incoming horizontal arc. Thus $|U| \le m$. Combining this inequality with Inequality (1), we obtain the desired result. □