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Sequential lottery contests with multiple participants

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HIGHLIGHTS

- We study a sequential lottery contest with multiple participants.
- We apply aggregative games techniques in a novel fashion.
- We show the existence of a unique subgame perfect equilibrium in pure strategies.

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ABSTRACT

We apply aggregative games techniques in a novel fashion in the analysis of sequential lottery contests with n players to show that, there exists a unique subgame perfect equilibrium in pure strategies.

1. Introduction

In an imperfectly discriminative contest, there is a probabilistic relation between players' investments and prize allocation. A lottery contest is a special yet commonly used imperfectly discriminative contest (Konrad, 2009).

The literature on imperfectly discriminative sequential contests has focused mainly on the two-player case (for example, see Dixit, 1987; Leininger, 1993; Morgan, 2003; Yildirim, 2005; Serena, 2017). Glazer and Hassin (2000) provided an analytical Stackelberg solution for a sequential lottery contest with three players; however they found it to be exceedingly difficult to obtain an analytic Stackelberg solution for more than three players.

We study analytically a sequential lottery contest (in which players choose their expenditures one by one and a lottery is held after the last player has made her choice of expenditure) with n players. This is a dynamic aggregative game. The "replacement correspondence technique associated with aggregative games is applied here in a novel fashion in order to show that in a sequential lottery contest with n players there exists a unique Subgame Perfect Equilibrium (henceforth: SPE) in pure strategies .[1]

2. The model

There are n identical risk neutral players in a sequential lottery contest in which the order of moves is exogenous. Each player $i \in N$ observing the effort made by previous players and anticipating the future expenditures of subsequent players, invests $x_i \geq 0$. A lottery with one winner and one prize with a common value $v = 1$ is held

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[1] Cornes and Hartley (2003, 2005) utilize the notion of replacement correspondence to study simultaneous contests. For definitions of aggregative games and replacement correspondence, see Cornes and Hartley (2012).

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after all the players have made their choice.[2] The probability of player $i \in N$ winning the prize is determined by the lottery contest success function:

$$p_i = \begin{Bmatrix} \frac{1}{n}, & \text{if } x_1 = x_2 \cdots = x_n = 0 \\ \frac{x_i}{\sum_{j=1}^{n} x_j}, & \text{otherwise} \end{Bmatrix}. \tag{1}$$

Define: $\forall i \neq 0 \;\; X_i = \sum_{j=1}^{i} x_j$, $X_0 = 0$, $X_n = X$ and $E\pi_i$ as player i's expected net payoff.

Each player i solves the following problem:

$$\max_{x_i} E\pi_i(x_i; X_{i-1}), \quad \text{where } E\pi_i(x_i; X_{i-1}) = \frac{x_i}{X_{i-1} + x_i + \sum_{j=i+1}^{n} x_j} - x_i. \tag{2}$$

In this game, player i's pure strategy is $x_i : [0, \infty) \to [0, \infty)$ for $i = 2, \ldots, n$, and given that investments are not made prior to the first play, the first player's strategy set is $x_1 \in [0, \infty)$. Thus, the sum of the players' strategies is $X \in [0, \infty)$ and an SPE in pure strategies is a strategy profile $\underline{x} = (x_1, \ldots, x_n)$ such that $x_i \in \arg\max_{x_i} E\pi_i(x_i; X_{i-1})$, where player i anticipates $x_{i+1}, \ldots, x_n \forall i \in N$.[3]

3. The alternative problem

3.1. The setting

In what follows, we define, by recursion, a function $X_i(X)$. It is used to present an alternative problem which is shown to be equivalent to the original problem.

The function $X_i(X)$ is defined as follows:

$$X_n(X) = X \tag{3}$$

and

$$X_{i-1}(X) = X_i(X) + X_i'(X^2 - X), \quad \text{where} \quad X_i' = \frac{dX_i(X)}{dX} \;\; \forall i \in \{2, \ldots, n\}. \tag{4}$$

Lemma 1 specifies the structure of the function $X_i(X)$.

Lemma 1. *$X_i(X) = \sum_{j=1}^{n-i+1} a_{ji} X^j \forall\, i = 1, \ldots, n$, where $a_{ji} \neq 0 \;\forall\, 2 \leq j \leq n$.*

The proofs of the lemmas and the propositions appear in the Appendix.

For a given X_{i-1}, $E\tilde{\pi}_i(X; X_{i-1})$ is defined as:

$$E\tilde{\pi}_i(X; X_{i-1}) = \frac{X_i(X) - X_{i-1}}{X} - (X_i(X) - X_{i-1}).\text{[4]} \tag{5}$$

The FOC for maximization of $E\tilde{\pi}_i$ with respect to X is:

$$\frac{\partial E\tilde{\pi}_i}{\partial X} = \frac{X_i'X + X_{i-1} - X_i(X)}{X^2} - X_i' = 0 \;\forall i. \tag{6}$$

Notice that by rearranging the terms in (6) we obtain (4)

Lemma 2 specifies the structure of the function $E\tilde{\pi}_i(X, 0)$.

[2] The assumption that $v = 1$ is made without loss of generality.

[3] Below we show that in each subgame of the original game there exists a unique SPE in pure strategies and thus each player can accurately anticipate the efforts made by subsequent players.

[4] Note that by definition, $X \geq X_{i-1}$.

Lemma 2. *$E\tilde{\pi}_i(X; 0)$ has the following structure:*

(i) $E\tilde{\pi}_i(X; 0)$ is a polynomial of degree $n-i+1$ with $n-i+1$ roots over the interval $[0, 1]$ and $n-i$ extrema in between them, which are the roots of $E\tilde{\pi}_{i-1}(X; 0)$.

(ii) Denote by X_{is}^0 and X_{il}^0 the smallest and largest roots, respectively, of $E\tilde{\pi}_i(X; 0)$ over the interval $(0, 1)$. Then, if $n - i$ is odd, $E\tilde{\pi}_i(X; 0)$ achieves a maximum over the interval $\left(0, X_{is}^0\right)$ as well as over the interval $\left(X_{il}^0, 1\right)$. If $n-i$ is positive and even, then $E\tilde{\pi}_i(X; 0)$ achieves a minimum over the interval $\left(0, X_{is}^0\right)$ and a maximum over the interval $\left(X_{il}^o, 1\right)$.

For example, $E\tilde{\pi}_{n-1}(X; 0) = X - X^2$ has two roots: $X = 0$ and $X = 1$ and a unique maximum at $X = 0.5$, where $E\tilde{\pi}_{n-2}(X; 0) = -2X^3 + 3X^2 - X$ has three roots: $X = 0$, $X = 0.5$ and $X = 1$, and two extrema: a minimum over the interval $(0, 0.5)$ and a maximum over the interval $(0.5, 1)$ and so on. The structure of $E\tilde{\pi}_i(X; 0)$ for $n - 4 \leq i \leq n - 1$ is illustrated in Fig. 1.

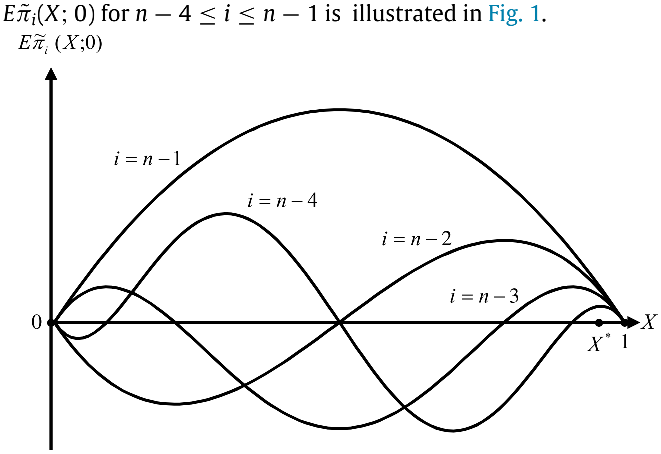

Fig. 1. The functions $E\tilde{\pi}_i(X; 0)$ over the interval $[0, 1]$ for $i = n - 4, \ldots, n - 1$.

3.2. The solution of the alternative problem

For a given $X_{i-1} < 1$, consider the following alternative problem.

$$\begin{aligned} \max_{X} \quad & E\tilde{\pi}_i(X; X_{i-1}) \\ \text{s.t.} \quad & X_i(X) - X_{i-1} \geq 0, \\ & X_j(X) - X_{j-1}(X) \geq 0 \;\forall j > i \\ & \text{and} \\ & X \leq 1. \end{aligned} \tag{7}$$

Proposition 1 utilizes both Lemma 1 and Lemma 2 to show that the alternative problem has a unique solution.

Proposition 1. *Problem (7) has a unique solution for all i, and it is an interior solution. Let X^* be the solution of (7) for $i = 1$. Then given that $X_{i-1} = X_{i-1}(X^*)$, X^* is also the solution of (7) for all $i > 1$.*

Notice that, the solution of (7) for $n = 5$ and $i = 1$, which is the unique maximum of $E\tilde{\pi}_1(X; 0)$ over the interval $\left(X_{1l}^o, 1\right)$, is illustrated in Fig. 1.

4. The solution of the original problem

We first present Lemma 3 since it is needed to prove our main result which follows it.

Lemma 3. *An SPE in pure strategies of the sequential lottery contest with n players satisfies: (i) $x_i > 0 \;\; \forall i \in N$ and (ii) $X = \sum_{j=1}^{n} x_j \in (0, 1)$.*

Proposition 2 demonstrates the link between the original and the alternative problem.