Games and Economic Behavior 110 (2018) 248–257

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Rationalizability and logical inference ☆

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ARTICLE INFO

Article history:
Received 27 January 2016
Available online xxxx

JEL classification:
C72

Keywords:
Rationalizability
Modal logic
Necessitation

ABSTRACT

In a model of modal logic it is shown that the assumptions of rationality and intelligence of the players imply that only rationalizable strategies can be played. Nothing more can be inferred from these rules. Hereby the assumption of "intelligence" expresses that whatever an outside observer can deduce about the play of the game can be inferred by the players themselves, provided they have the same information. In our framework the assumption of intelligence is simply the familiar inference rule of necessitation in modal logic. Our approach contrasts with a hierarchical approach traditional in the literature, where assumption about knowledge about knowledge ... about rationality are added one by one.

1. Introduction

This paper provides an alternative justification of rationalizability where logical inference and the intelligence of the players are central. The basic question is, as in the seminal papers by Bernheim (1984) and Pearce (1984), "...the fundamental problem of what can be inferred about the outcome of a non-cooperative game, from the rationality of the players and the information they possess..." (Pearce, 1984). The word "inference" does hereby not solely refer to what a game theorist can deduce, but also to what the players themselves can infer. This becomes apparent by a further citation from Pearce (1984): "the most sweeping [...] case for Nash equilibrium [...] asserts that a player's strategy must be a best response to those selected by the other players, because he can *deduce* what these strategies are." Classical game theory has always been willing to make strong assumptions on what players can infer. Hence one often finds discussed what Myerson (1991) has coined the assumption of the INTELLIGENCE of the players. By this he means the assumption that whatever an outsider can infer about a game can also be inferred by the players themselves, if they have access to the same information. The assumption of intelligence is certainly very stringent and not always behaviorally correct. However, for applications in business and politics it may often be wiser to assume that the opponents are cleverer than you are yourself rather than the other way around, and this is why the assumption is so appealing for game theory.

In this paper we show in a simple syntactic model of propositional modal logic that the assumptions of rationality and intelligence of the players imply that only rationalizable strategies can be played. We also show that nothing further can be deduced from these assumptions.

☆ The paper was submitted for the Special Issue In Honor of John Nash and should be regarded as part of that volume.
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[1] The motivation for this paper goes back many years, when Amanda Friedenberg questioned the role of monotonicity for rationalizability concepts, which I had always presumed. Presentations in Singapore, Exeter, Bielefeld, Maastricht, Jerusalem, Bath, Oxford and York, the comments received on these occasions and the comments received from the referees were essential to clarify my ideas. I am grateful for all these suggestions.

https://doi.org/10.1016/j.geb.2018.04.006
0899-8256/

Hereby "intelligence" is taken as the inference rule known as necessitation in modal logic. In our context, if a statement A in the logic is provable then $\vdash_i A$, holds, where $\vdash_i$ is the modal operator to express "player i can infer statement A". The assumption is frequently used in epistemic game theory with reference to knowledge and related concepts, for instance in the work of Aumann, e.g., in Aumann (1999)[2] and Arieli and Aumann (2015). The only surprising bit is the importance this assumption plays in our model.

The modal logic we are using is a very basic extension of propositional logic. The only reason to work with a modal logic is that we cannot express in a straightforward way within propositional logic that a statement from propositional logic can be proved. Modal logic allows us to distinguish between accidental and necessary truth (see Fitting and Mendelsohn, 1998, page 2). One of the major uses of modal operators, for instance in model theory, is hence to express within a logical system what can be proved about the system. This is exactly how we are using the modal operators. Thereby we do not aim for completeness, our main result only concerns the provability of certain simple propositional statements. Our approach is much simpler than, e.g., Kaneko and Nagashima (1996); Hu and Kaneko (2015) or Bjorndahl et al. (2017). In fact, as a referee noticed, with a suitable inference rule our result can be obtained within a purely propositional logic.

Our approach is syntactic, i.e., we use logical statements, axioms and inference rules which show explicitly what deductions are made, by the outside observer and by the players. The traditional approach using knowledge or beliefs is typically semantic and uses state space models, if it is formalized at all. In the textbook approach, for instance in Maschler et al. (2015), it is first assumed that all players are rational given that they know that the game is played. Next it is assumed that every player knows this and is rational given this knowledge. Since this is not enough, it is assumed in addition that all players know that all players are rational, and so on. In this approach the assumption of intelligence is at best implicit. It is not clear whether each new assumption is added by the modeler because it is plausible or whether it follows logically because the players are rational and intelligent. The semantic approach, as, for instance, discussed in Perea (2012), based on models using knowledge partitions on a state space or similar, does not make this immediately visible either.

It is crucial for our approach that we do not use any monotonicity assumptions, neither in our formulation of rationality nor for the inference operators. However, as we shall see, the interplay of the inference rules of intelligence and modus ponens will have strong monotonicity implications.

Concerning rationality, we describe it by a very general operator which allows for many different interpretations, including straightforward irrationality. In the most standard classical interpretation of Bayesian rationality in games, where players take ex ante optimal choices given their ex ante beliefs about the opponent's strategy choices, our approach yields the standard notion of rationalizability. When applying our approach to some "refined" notions of rationality as in Börgers (1994) and Brandenburger (1992), we obtain in two-player games variations of the Dekel/Fudenberg procedure, where in the first round strategies which are not "refined" best replies are eliminated and thereafter iteratedly all strongly dominated strategies are eliminated (see also Balkenborg et al., 2015). An elimination process which also fits with a slight adjustment of our framework is the iterated elimination of flaws introduced by Stalnaker (1994). It is further analyzed by Hillas and Samet (2015a) and Hillas and Samet (2015b) who show that it yields the maximal non-probabilistic correlated equilibrium.

Our approach can also be used to express that players do not use strategies which are weakly dominated relative to the strategies they have inferred as possible. Then, however, our logic may be contradictory if the iterated elimination of weakly dominated strategies leads to inconsistencies. Our approach does hence not solve the problems addressed, for instance, in Samuelson (1992).

Our approach imposes rationality *whenever meaningfully defined*. This has strong monotonicity implications and can hence not capture the formulae describing common assumption of rationality in Brandenburger et al. (2008) or common strong belief in rationality as in Battigalli and Siniscalchi (2002) or Arieli and Aumann (2015). These carefully constructed iterative formulae avoid contradictions stemming from non-monotonicity by imposing rationality only *when needed* according to the respective formula.

I believe that our syntactic model, where we can deduce rationalizability from rationality and intelligence, is not only straightforward and natural. It also helps to understand better the role monotonicity assumptions have or do not have in a satisfactory formulation of rationalizability and its refinements.

The paper is organized as follows. After the introduction we introduce the game logic in Section 2, first the well-defined formulae, then the axioms and inference rules. We also introduce the notion of the lower monotone hull of an operator on sets, which is crucial for stating and proving our main result in Section 3. This main result does not provide existence of rationalizable strategies. A very limited existence result based on monotonicity is hence also provided in Section 3. Section 4 gives a quick overview of what our model yields for a variety of notions of rationality that have appeared in the literature and provides slight extensions of our basic framework. Section 5 concludes.

[2] See the discussion around requirement (4.3) in the paper.