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Decomposition of solutions and the Shapley value ☆

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ABSTRACT

We suggest foundations for the Shapley value and for the naïve solution, which assigns to any player the difference between the worth of the grand coalition and its worth after this player left the game. To this end, we introduce the decomposition of solutions for cooperative games with transferable utility. A decomposer of a solution is another solution that splits the former into a direct part and an indirect part. While the direct part (the decomposer) measures a player's contribution in a game as such, the indirect part indicates how she affects the other players' direct contributions by leaving the game. The Shapley value turns out to be unique decomposable decomposer of the naïve solution.

1. Introduction

The Shapley value (Shapley, 1953) is probably the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). Hart and Mas-Colell (1989) suggest an indirect characterization as marginal contributions of a potential (function).[1] Roth (1977) shows that the Shapley value can be understood as a von Neumann–Morgenstern utility. As a contribution to the Nash program, which aims at building bridges between cooperative and non-cooperative game theory, Pérez-Castrillo and Wettstein (2001) implement the Shapley value as the outcomes of the sub-game perfect equilibria of a combined bidding and proposing mechanism, which is modeled by a non-cooperative extensive form game.[2]

Among the solution concepts for TU games, the Shapley value can be viewed as *the* measure of the players' *own* productivity in a game. This view is strongly supported by Young's (1985) characterization by three properties: efficiency, strong monotonicity, and symmetry. Efficiency says that the worth generated by the grand coalition is distributed among

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[1] Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.

[2] Ju and Wettstein (2009) suggest a class of bidding mechanisms that implement several solution concepts for TU games.

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the players. Strong monotonicity requires a player's payoff to increase weakly whenever *her* productivity, measured by *her* marginal contributions to *all coalitions* of the other players, weakly increases. Symmetry ensures that equally productive players obtain the same payoff.

A perhaps naïve way to measure a particular player's productivity within a game is to only look at the marginal contribution of this player to the coalition of *all others*, which we address as the "naïve solution". This solution, however, is problematic for (at least) two reasons. First, in general, the naïve payoffs do not sum up to the worth generated by the grand coalition. Hart and Mas-Colell (1989) use this fact as a motivation of the potential approach to the Shapley value. Second and not less important, one could argue that every player's presence is necessary for generating the naïve payoff of any given player, and therefore the productivity gains reflected in that payoff should be partly attributed to the others.

In order to tackle the second problem of the naïve solution mentioned above, we suggest the decomposition of solutions. A solution ψ decomposes a solution φ if it splits φ into direct and indirect contributions in the following sense. A particular player's payoff for φ is the sum of her payoff for ψ (direct contribution) and what the other players gain or lose under ψ when this particular player leaves the game (indirect contribution). That is, the indirect contributions reflect what a player contributes to the other players' direct contributions. We say that a solution is decomposable if there exists a decomposer, i.e., a solution that decomposes it.

We show that the Shapley value is the unique decomposable decomposer of the naïve solution (Theorem 3). The naïve solution thus conveys interesting information about the Shapley value; and its decomposition may be viewed as a rationale for the naïve solution in terms of the Shapley value. Vice versa, the Shapley value emerges as the natural decomposition of a player's marginal contribution to all other players in the sense that the Shapley value itself can be further rationalized in terms of some underlying solution.

We answer the question of which solutions are decomposable by showing that decomposability is equivalent to a number of other well-known properties of solutions: balanced contributions (Myerson, 1980), path independence (Hart and Mas-Colell, 1989), consistency with the Shapley value (Calvo and Santos, 1997), and admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998) (Theorem 4).

We further establish that amongst all the decomposers of a decomposable solution, there is one and only one that is itself decomposable. It follows immediately that, starting with any decomposable solution φ, there is a unique sequence in which each term is a decomposer of its predecessor (Theorem 7). We call this sequence the "resolution" of φ. Resolutions allow us to capture higher-order contributions, where for instance the third-order contribution captures what player i contributes to player j's contribution to player k's payoff. We explore the structure of higher-order decomposers using higher-order contributions (Theorem 12).

The remainder of this paper is organized as follows. In Section 2, we give basic definitions and notation. In the third section, we formalize and study the notions of decomposition and decomposability outlined above and present our new rationale for the naïve solution and the Shapley value. The fourth section investigates the notion of decomposability. The fifth section relates higher-order decompositions to higher-order contributions. Some remarks conclude the paper. All proofs are contained in the appendices.

2. Basic definitions and notation

A (TU) game on a finite player set N is given by a **characteristic function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. The set of all games on N is denoted by $\mathbb{V}(N)$. Let $\mathcal{N}$ denote the set of all finite player sets.[3] The cardinalities of $S, T, N, M \in \mathcal{N}$ are denoted by s, t, n, and m, respectively.

For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}(N)$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. As pointed out in Shapley (1953), these unanimity games form a basis of the vector space[4] $\mathbb{V}(N)$, i.e., any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{1}$$

where the **Harsanyi dividends** $\lambda_T(v)$ can be determined recursively via $\lambda_T(v) = v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v)$ for all $T \subseteq N$, $T \neq \emptyset$ (see Harsanyi, 1959).

A **solution/value** is an operator φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$. The **Shapley value** (Shapley, 1953) distributes the dividends $\lambda_T(v)$ equally among the players in T, i.e.,

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t} \tag{2}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. A solution is **efficient** if $\sum_{i \in N} \varphi_i(v) = v(N)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$.

[3] We assume that the player sets are subsets of some given countably infinite set $\mathfrak{U}$, the universe of players; $\mathcal{N}$ denotes the set of all finite subsets of $\mathfrak{U}$.

[4] For $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.

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