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The Stochastic Shapley Value for coalitional games with externalities

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ABSTRACT

A long debated but still open question in the game theory literature is that of how to extend the Shapley Value to coalitional games with externalities. While previous work predominantly focused on developing alternative axiomatizations, in this article we propose a novel approach which centers around the coalition formation process and the underlying probability distribution from which a suitable axiomatization naturally follows. Specifically, we view coalition formation in games with externalities as a *discrete-time stochastic process*. We focus, in particular, on the Chinese Restaurant Process – a well-known stochastic process from probability theory. We show that reformulating Shapley's coalition formation process as the Chinese Restaurant Process yields in games with externalities a unique value with various desirable properties. We then generalize this result by proving that all values that satisfy the direct translation of Shapley's axioms to games with externalities can be obtained using our approach based on stochastic processes.

1. Introduction

The problem of how to fairly divide a value obtained through cooperation is one of the most fundamental issues studied in coalitional game theory. It is relevant to a wide range of economic and social situations, from sharing the cost of a local wastewater treatment plant, through dividing the annual profit of a joint venture enterprise, to determining power in voting bodies. Assuming that the coalition of all the players (*i.e.* the *grand coalition*) forms, Shapley (1953) proposed to remunerate the players by considering their marginal contributions to all coalitions they could possibly join. His division scheme, now called the Shapley Value, is the unique one satisfying the following four axioms: *Efficiency* – all payoff is distributed among the players; *Null-Player* – a player with no influence on payoffs receives nothing; *Symmetry* – symmetric players obtain the same payoff; and *Additivity* – the division scheme is additive over games.

However, Shapley's remarkable result holds only when one cooperative arrangement of players does not impose any externalities on any other cooperative arrangements. Such an assumption is clearly untenable in many practical economic situations of interest. For example, in an oligopolistic market, joint R&D projects are meant to increase the competitive edge of cooperating companies (Yi, 2003). Similarly, the extent of pollution reduction achieved by an international treaty depends not only on the signatories to the treaty, but also on similar agreements among non-participants (Finus, 2003). In all such situations it is necessary to consider coalitions not as independent entities but rather coalitions embedded in

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coalition structures. Extending the Shapley Value to this richer setting has been a subject of ongoing debate in the literature for more than forty years. This issue is also the focus of our article.

Mathematically, the Shapley Value in games with no externalities is defined as the average marginal contribution of a player, taken over all possible permutations of players. It can be rationalized by the following *deterministic coalition formation process* proposed by Shapley (1953, p. 39). Assume that players create the grand coalition sequentially in a random order. A new player receives the payoff that equals his marginal contribution to the group of players that he joins. Now, the Shapley Value is the player's expected such payoff over all random orders (permutations) of players.

Extending the Shapley Value has turned out to be a challenging research problem because an obvious (or, as we will say, *direct*) *translation* of Shapley's axioms to games with externalities does not yield a unique value. The key problem is the Null-Player Axiom, as its direct translation to games with externalities is too weak to guarantee uniqueness.

A number of methods have been developed in the literature to address this issue. Some, such as Pham Do and Norde (2007) and Hu and Yang (2010), obtained uniqueness by proposing a stronger version of the original Null-Player Axiom. Other contributors have moved increasingly further away from Shapley's original axiomatization by adding new axioms, and sometimes dropping some of the original ones. An example of such an approach is the work by Grabisch and Funaki (2008), who used Markovian and Ergodic Axioms and modified the Symmetry and the Null-Player Axioms. Yet another method has been to build extensions to games with externalities relying on alternative axiomatizations of the original Shapley Value, such as Myerson's (1980) balanced-contribution axiomatization or Young's (1985) monotonicity axiomatization.

While previous works predominantly focused on axiomatizations for games with externalities, the extension of the Shapley Value proposed in this article centers around the *coalition formation process* and the underlying probability distribution from which a suitable axiomatization naturally follows. In particular, we view coalition formation in games with externalities as a *discrete-time stochastic process*, in which the players leave the grand coalition one after another in a random order. Next, they either join one of the existing groups outside or form a completely new group, all with a certain probability. Each player receives the payoff that equals her marginal contribution to the coalition she left.

One of the fundamental stochastic processes studied in the literature is the Chinese Restaurant Process. This process and its variants have been used in a variety of applications ranging from modeling texts to genetics and functional genomics (Neal, 2000; Blei et al., 2004; Qin, 2006; Xing et al., 2007; Ewens, 1972). As we will show in this article, the Chinese Restaurant Process turns out to be of special interest also in the context of coalitional games with externalities. To gain some intuition behind it, let us consider the following fictional scenario. Let us imagine a Chinese restaurant with an infinite number of tables, each with an infinite number of seats. The first customer sits at the first table. Every new customer chooses a seat next to a customer that sits at one of the occupied tables, or a seat at the first unoccupied table, always with the same probability. Hence, the k-th customer chooses a table with b customers with probability $\frac{b}{k}$ or chooses an unoccupied table with probability $\frac{1}{k}$. In our cooperative game context, each table represents a coalition of players and all the tables taken together represent a coalition structure that emerged as a result of a stochastic coalition formation process.

Our key results can be summarized as follows. We show that reformulating Shapley's coalition formation process as the Chinese Restaurant Process yields a unique value in games with externalities that has various desirable properties. In particular, it is a well-known value from the literature derived from two different axiomatizations: first, by Feldman (1996) using axioms of Additivity, Symmetry, Carrier (which is a combination of the Null-Player Axiom and Efficiency), Per Capita Liability, and Per Capita Claim; next, by Macho-Stadler et al. (2007) using the axioms of Efficiency, Null-Player, Additivity, Strong Symmetry and Similar Influence.[1] In this article, using our stochastic process approach, we show that this value can be derived from the direct translation of Shapley's axioms to games with externalities, where we obtain uniqueness by strengthening the original Null-Player Axiom based on the Chinese Restaurant Process. In what follows, we refer to this value as the *Stochastic Shapley Value*.

Next, we extend the above result by considering a generalized version of the Chinese Restaurant Process in which the partition of players emerges with an arbitrary probability distribution. We prove that all values that satisfy the direct translation of Shapley's axioms to games with externalities can be obtained using such a generalized process. This means, in particular, that by choosing appropriate probability distributions we obtain the values by Pham Do and Norde (2007), McQuillin (2009), Bolger (1989), Hu and Yang (2010), and Myerson (1977). We also generalize the axiomatization by proving that the direct translation of Shapley's axiomatization – Efficiency, Symmetry, Additivity and the Null-Player Axiom strengthened using the generalized stochastic process – is enough to obtain uniqueness. This result yields, in particular, axiomatizations of these five values that are close to Shapley's original axiomatization.

The remainder of this article is organized as follows. In the next section, we present basic definitions and notation. In Section 3, building upon on the Chinese Restaurant Process, we derive and then axiomatize the Stochastic Shapley Value. In Section 4, we consider the generalization of this approach. Finally, in Section 5, we position our results among other works in the literature. Conclusions follow.

[1] See Section 5 for more details on both axiomatizations.